# PRIMITIVE VECTORS OF KAC-MODULES OF THE LIE SUPERALGEBRAS $sl(m/n)$

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**ABSTRACT.** To study finite-dimensional modules of the Lie superalgebras, Kac introduced the Kac-modules $\overline{V}(\Lambda)$ and divided them into typical or atypical modules according as they are simple or not. For $\Lambda$ being atypical, Hughes *et al* have an algorithm to determine all the composition factors of a Kac-module; they conjectured that there exists a bijection between the composition factors of a Kac-module and the so-called permissible codes. The aim of this paper is to prove this conjecture. We gives a partial proof here, i.e., to any unlinked code, by constructing explicitly the primitive vector, we prove that there corresponds a composition factor of the Kac-module. We will give a full proof of the conjecture in another paper.

**KEYWORDS:** Kac-module, atypical, composition factor, primitive weight, code.

## I. INTRODUCTION

In the classification of finite-dimensional modules of the basic classical Lie superalgebras,[3−6,10] Kac distinguished between typical and atypical modules. He also introduced now the so-called Kac-module $\overline{V}(\Lambda)$, which was shown to be simple if and only if $\Lambda$ is typical. For $\Lambda$ being atypical, the problem of the structure of $\overline{V}(\Lambda)$, or equivalently, the character of the simple module $V(\Lambda)$, has been the subject of intensive study.[2,11,14−16] More generally, the problem of classifying indecomposable modules has received much attention in the literature.[1,8,9,11,12] Kac obtained a character formula for typical modules.[5] The problem for atypical $sl(m/n)$-modules has seemed to be difficult, though several partial solutions have been achieved.[2,15,16]

Only recently Serganova[11] found a solution for the characters of simple $gl(m/n)$-modules, who described the multiplicities $a_{\Lambda\Sigma}$ of composition factors $V(\Sigma)$ of $\overline{V}(\Lambda)$ in terms of Kazhdan-Lusztig polynomials. However, Serganova's algorithm of describing $a_{\Lambda\Sigma}$ turns out to

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be rather complicated. The structure of $\overline{V}(\Lambda)$ is still not so apparent to readers. Hughes *et al*[2] derived an algorithm to determine all the composition factors of $sl(m/n)$-Kac-modules $\overline{V}(\Lambda)$. They conjectured that there exists a bijection between the composition factors of $\overline{V}(\Lambda)$ and the permissible codes (Definition 3.9). This conjecture clearly describes the structure of $\overline{V}(\Lambda)$. The aim of the present paper and the forthcoming paper[13] is to prove this conjecture. In this paper, we prove that to any unlinked code, there corresponds a composition factor of $\overline{V}(\Lambda)$, by constructing explicitly a primitive vector corresponding to the unlinked code (Theorems 6.6&6.12). Then in *Ref.* 13, we will give a full proof of the conjecture and point out that the proof of the conjecture will result in the proofs of some other conjectures.

## II. THE LIE SUPERALGEBRA $sl(m+1/n+1)$

Denote $G{=}sl(m{+}1/n{+}1)$ the set of $(m{+}n{+}2){\times}(m{+}n{+}2)$ matrices $x{=}\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)$ of zero supertrace $str(x){=}tr(A){-}tr(D){=}0$, where $A,B,C,D$ are $(m{+}1){\times}(m{+}1)$,$(m{+}1){\times}(n{+}1)$,$(n{+}1){\times}(m{+}1)$, $(n{+}1){\times}(n{+}1)$ matrices respectively. Let $G_{\overline{0}}{=}\{\left(\begin{smallmatrix}A & 0\\ 0 & D\end{smallmatrix}\right)\}$, $G_{\overline{1}}{=}\{\left(\begin{smallmatrix}0 & B\\ C & 0\end{smallmatrix}\right)\}$, then $G{=}G_{\overline{0}}{\oplus}G_{\overline{1}}$ as a $\mathbb{Z}_2$ $=\mathbb{Z}/2\mathbb{Z}$ graded space, is a Lie superalgebra with bracket: $[x,y]{=}xy{-}(-1)^{\xi\eta}yx$ for $x{\in}G_\xi$, $y{\in}G_\eta, \xi,\eta{\in}\mathbb{Z}_2$ such that $G_{\overline{0}}{\cong}sl(m{+}1){\oplus}\mathbb{C}{\oplus}sl(n{+}1)$ is a Lie algebra. Let $G_{+1}{=}\{\left(\begin{smallmatrix}0 & B\\ 0 & 0\end{smallmatrix}\right)\}, G_{-1}{=}$ $\{\left(\begin{smallmatrix}0 & 0\\ C & 0\end{smallmatrix}\right)\}$. Then $G$ has a $\mathbb{Z}_2$-consistent $\mathbb{Z}$ grading $G{=}G_{-1}{\oplus}G_0{\oplus}G_{+1}, G_{\overline{0}}{=}G_0$, $G_{\overline{1}}{=}G_{-1}{\oplus}G_{+1}$.

The Cartan subalgebra $H$ consisting of diagonal $(m{+}n{+}2){\times}(m{+}n{+}2)$ matrices of zero supertrace has dimension $m{+}n{+}1$. The weight space $H^*$ is the dual of $H$, spanned by the forms $\epsilon_a$ $(a{=}1,\ldots,m{+}1)$, $\delta_b$ $(b{=}1,\ldots,n{+}1)$, where $\epsilon_a$: $x{\to}A_{aa}$, $\delta_b$: $x{\to}D_{bb}$ for $x{=}\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)$, with $\sum_{a=1}^{m+1}\epsilon_a{-}\sum_{b=1}^{n+1}\delta_b{=}0$; it has an inner product derived from the Killing form that $\langle\epsilon_a|\epsilon_b\rangle{=}\delta_{ab}$, $\langle\epsilon_a|\delta_b\rangle{=}0$, $\langle\delta_a|\delta_b\rangle{=}{-}\delta_{ab}$, where $\delta_{ab}$ is the Kronecker symbol. Let $\Delta,\Delta_0,\Delta_1$ be sets of roots, even, odd roots respectively, $e(\alpha)$ the root vector corresponding to $\alpha$. $G$ has a root space decomposition $G{=}H{\oplus}\oplus_{\alpha\in\Delta}\mathbb{C}\,e(\alpha)$ with the roots and root vectors given by

$$
\begin{array}{lcll}
\epsilon_a-\epsilon_b & \leftrightarrow & E_{ab} & (1\leq a,b\leq m{+}1, a{\neq}b) \quad \text{(even)},\\
\delta_a-\delta_b & \leftrightarrow & E_{m+a+1,m+b+1} & (1\leq a,b\leq n{+}1, a{\neq}b) \quad \text{(even)},\\
\epsilon_a-\delta_b & \leftrightarrow & E_{a,m+b+1} & (1\leq a\leq m{+}1, 1\leq b\leq n{+}1) \quad \text{(odd)},\\
\delta_a-\epsilon_b & \leftrightarrow & E_{m+a+1,b} & (1\leq a\leq n{+}1, 1\leq b\leq m{+}1) \quad \text{(odd)},
\end{array}
$$

where $E_{ab}$ is the matrix with entry 1 at $(a,b)$ and 0 otherwise. We shall find it convenient to use a notation for roots somewhat different to that in previous papers.[2,15,16] Define sets

$$I_1{=}\{\overline{m},...,\overline{1}\},\ I_2={\{1,...,n\}},\ I{=}I_1\cup\{0\}{\cup}I_2, \text{ where } \overline{i}{=}{-}i, i{\in}\mathbb{Z}_+.$$

Choose a basis for $H$: $h_i{=}E_{m+i+1,m+i+1}{-}E_{m+i+2,m+i+2}, i{\in}I_1{\cup}I_2, h_0{=}E_{m+1,m+1}{+}E_{m+2,m+2}$. The simple roots in $H^*$ are: $\alpha_i{=}\epsilon_{m+i+1}{-}\epsilon_{m+i+2}, i{\in}I_1, \alpha_0{=}\epsilon_{m+1}{-}\delta_1, \alpha_i{=}\delta_i{-}\delta_{i+1}, i{\in}I_2$. Thus $\alpha_0$ is the only odd simple root. The corresponding Dynkin diagram is

$$
\begin{array}{ccccccccc}
\circ\!\!-\!\!- & \circ\!\!-\!\!-\cdots\cdots\!\!-\!\!- & \circ\!\!-\!\!- & \otimes\!\!-\!\!- & \circ\!\!-\!\!-\cdots\!\!-\!\!- & \circ\!\!-\!\!- & \circ\\
\alpha_{\overline{m}} & \alpha_{\overline{m-1}} & \alpha_{\overline{1}} & \alpha_0 & \alpha_1 & \alpha_{n-1} & \alpha_n
\end{array}
\tag{2.1}
$$

with $I_1,I_2$ corresponding to $sl(m{+}1), sl(n{+}1)$. The symmetric inner product satisfies

$$
\begin{array}{lll}
\langle\alpha_i|\alpha_i\rangle=2, i\in I_1, & \langle\alpha_0|\alpha_0\rangle=0, & \langle\alpha_i|\alpha_i\rangle=-2, i\in I_2,\\
\langle\alpha_{i-1}|\alpha_i\rangle=-1, i\in I_1, & \langle\alpha_0|\alpha_{\pm1}\rangle=\pm1, & \langle\alpha_i|\alpha_{i+1}\rangle=1, i\in I_2,
\end{array}
\tag{2.2}
$$

and $\langle\alpha_i|\alpha_j\rangle{=}0, j{\neq}i, i{\pm}1$ and $h_i(\alpha_j){=}\alpha_j(h_i){=}\langle\alpha_i|\alpha_j\rangle, i{\leq}0$ or $-\langle\alpha_i|\alpha_j\rangle, i{>}0$. Define

$$\lambda,\mu\in H^*:\lambda\geq\mu\Leftrightarrow\lambda-\mu=\sum_{i\in I}k_i\alpha_i \text{ with all } k_i\geq 0, \tag{2.3}$$

a partially order on $H^*$. Let $\Delta^{\pm}(\Delta_{\overline{0}}^{\pm},\Delta_{\overline{1}}^{\pm})$ be sets of positive/negative roots (even, odd roots). Elements of $\Delta^+$ are sums of simple roots corresponding to connected subdiagrams of (2.1). Let $\alpha_{ij}{=}\sum_{k=i}^{j}\alpha_k$, then $\Delta_{\overline{0}}^{\pm}{=}\{\pm\alpha_{ij}|i{\leq}j, i{,}j{\in}I_1 \text{ or } i{,}j{\in}I_2\}$, $\Delta_{\overline{1}}^{\pm}{=}\{\pm\alpha_{ij}|i{\in}I_1{\cup}\{0\}{,}j{\in}\{0\}{\cup}I_2\}$. The root vectors $e_{ij}{=}e(\alpha_{ij}), f_{ij}{=}f(\alpha_{ij}){=}e(-\alpha_{ij})$ and the elements $h_{ij}$ of $H$ are

$$e_{ij}{=}E_{m+i+1,m+j+2},\ f_{ij}{=}E_{m+j+2,m+i+1},\ h_{ij}{=}E_{m+i+1,m+i+1}-(-1)^{\sigma_{ij}}E_{m+j+2,m+j+2},$$

where $\sigma_{ij}{=}0$ or $1\Leftrightarrow\alpha_{ij}$ is even or odd. Set $e_i{=}e_{ii}, f_i{=}f_{ii}$. The above implies $h_i{=}h_{ii}$ and

$$h_{ij}=\sum_{k=i}^{j}h_k,\ i,j\in I_1 \text{ or } i,j\in I_2, \quad\text{and}\quad h_{ij}=\sum_{k=i}^{0}h_k-\sum_{k=1}^{j}h_k,\ i\leq 0, j\geq 0.$$

The set $\{e_{ij},f_{ij},h_i|i,j\in I, i\leq j\}$ yields a basis for $G$, with the following nontrivial relations:

$$\begin{aligned}
&[e_{ij},e_{j+1,\ell}]=e_{i\ell},\quad [f_{ij},f_{j+1,\ell}]=-f_{i\ell},\quad [e_{ij},f_{ij}]=h_{ij},\\
&[e_{ij},f_{ik}]=\begin{cases}-(-1)^{\sigma_{ij}\sigma_{ik}}f_{j+1,k} \text{ if } j<k,\\ -(-1)^{\sigma_{ij}\sigma_{ik}}e_{k+1,j} \text{ if } j>k,\end{cases}\quad [e_{ik},f_{jk}]=\begin{cases}e_{i,j-1} \text{ if } i<j,\\ f_{j,i-1} \text{ if } i>j,\end{cases}\\
&[h_{ij},e_{k\ell}]{=}\mu e_{k\ell}, [h_{ij},f_{k\ell}]{=}-\mu f_{k\ell}, \mu{=}\delta_{i,k}-\delta_{i,\ell+1}-(-1)^{\sigma_{ij}}\delta_{j,k-1}+(-1)^{\sigma_{ij}}\delta_{j,\ell}\,.
\end{aligned} \tag{2.4}$$

Set $G_{\overline{0}}^{\pm}=\text{span}\{e(\alpha)|\alpha\in\Delta_{\overline{0}}^{\pm}\}, G\pm_1=\text{span}\{e(\beta)|\beta\in\Delta_{\overline{1}}^{\pm}\}, G^{\pm}=G_{\overline{0}}^{\pm}\oplus G_{\overline{1}}^{\pm}$. Note that $G_{\overline{1}}^{\pm}=G_{\pm 1}, G_{\overline{0}}=G_{\overline{0}}^{-}\oplus H\oplus G_{\overline{0}}^{+}, G=G^{-}\oplus H\oplus G^{+}$. Let $\mathbf{U}(G)$ be the universal enveloping algebra of $G$, $\mathbf{U}(G')$ that of its subalgebras $G'$ which is $H$-diagonalizable. Denote by $\mathbf{U}(G')_\eta$ the subspace of weight $\eta$. The PBW theorem can be extended to Lie superalgebras:[4,7]

**Theorem 2.1.** Let $y_1,\cdots,y_M$ be a basis of $G_{\overline{0}}$ and $z_1,\cdots,z_N$ be that of $G_{\overline{1}}$. The elements of the form $(y_1)^{k_1}\cdots(y_M)^{k_M}z_{i_1}\cdots z_{i_s}$, where $k_i\geq 0$ and $1\leq i_1<\cdots<i_s\leq N$, form a basis of $\mathbf{U}(G)$. ∎

For $\lambda\in H^*$, define its *Dynkin labels* to be $a_i=\lambda(h_i), i\in I$. These uniquely determine $\lambda$, which can then be represented as $\lambda=[a_{\overline{m}},\cdots,a_{\overline{1}};a_0;a_1,\cdots,a_n]$. $\lambda$ is called *dominant* if $a_i\geq 0$ for all $i\neq 0$, *integral* if $a_i\in\mathbb{Z}$ for all $i\neq 0$. The following convention will be useful later.

*Convention 2.2.* If $\Gamma$ denotes any quantity relating to $G=sl(m+1/n+1)$, then $\Gamma^{(m'/n')}$ denotes the same quantity relating to $sl(m'+1/n'+1)$. Thus $\Gamma^{(m/n)}=\Gamma$. ∎

## III. THE KAC-MODULES

Let $V^0(\Lambda)$ be the simple $G_{\overline{0}}$-module with integral dominant highest weight $\Lambda$ and vector $v_\Lambda$. Extend $V^0(\Lambda)$ to be a $G_{\overline{0}}\oplus G_{+1}$ module by setting $G_{+1}V^0(\Lambda)=0$. The *Kac-module*[6] is

$$\overline{V}(\Lambda)=\text{Ind}_{G_0\oplus G_{+1}}^{G}V^0(\Lambda)=\mathbf{U}(G)\otimes_{G_0\oplus G_{+1}}V^0(\Lambda).$$

By Theorem 2.1, $\mathbf{U}(G)=\mathbf{U}(G_{-1})\otimes\mathbf{U}(G_0)\otimes\mathbf{U}(G_{+1})$. It implies $\overline{V}(\Lambda)\cong\mathbf{U}(G_{-1})\otimes V^0(\Lambda)$. We summarize some well known properties of $\overline{V}(\Lambda)$; more details can be found in *Refs.* 2, 6. By definition, it is a $2^{(m+1)(n+1)}\text{dim}V^0(\Lambda)$ dimensional highest weight module generated by the

highest weight vector $v_\Lambda$, indecomposable and $H$-diagonalizable and it contains a maximal submodule $M = \{v \in \overline{V}(\Lambda) | v_\Lambda \notin \mathbf{U}(G)v\}$, such that $V(\Lambda) = \overline{V}(\Lambda)/M$ is a finite-dimensional simple module with highest weight $\Lambda$. Define $\rho = \rho_0 - \rho_1, \rho_0 = \frac{1}{2}\sum_{\alpha\in\Delta_0^+}\alpha, \rho_1 = \frac{1}{2}\sum_{\beta\in\Delta_1^+}\beta$.

*Definition 3.2.* $\Lambda, \overline{V}(\Lambda), V(\Lambda)$ are called *typical* if $\langle\Lambda+\rho|\beta\rangle \neq 0$ for all $\beta \in \Delta_1^+$. If $\beta \in \Delta_1^+$ such that $\langle\Lambda+\rho|\beta\rangle = 0$, then $\Lambda, \overline{V}(\Lambda), V(\Lambda)$ are called *atypical* and $\beta$ is an *atypical root* for $\Lambda$. If there exist precisely $r$ distinct atypical roots for $\Lambda$, we call $\Lambda, \overline{V}(\Lambda), V(\Lambda)$ *r-fold atypical.* ∎

**Theorem 3.3 (Kac[6]).** Every finite-dimensional simple $G$-module is isomorphic to a $V(\Lambda)$, characterized by its integral dominant highest weight $\Lambda$. $\overline{V}(\Lambda)$ is simple $\Leftrightarrow$ $\Lambda$ is typical. ∎

A *composition series* of $\overline{V}(\Lambda)$ is a sequence $\overline{V}(\Lambda) = V_0 \supset V_1 \supset \cdots$ with each $V_i/V_{i+1}$ isomorphic to some simple module $V(\Sigma)$, called a *composition factor* of $\overline{V}(\Lambda)$. A conjecture was made in *Ref.* 2, giving all the composition factors of $\overline{V}(\Lambda)$. We aim to prove the existence of some of these composition factors; for this, important concepts are those defined as follows.

*Definition 3.4.* A vector $v$ in a $G$-module $V$ is called *weakly G-primitive* if there exists a $G$-submodule $U$ of $V$ such that $v \notin U$ and $G^+v \subset U$. If $U = 0$, $v$ is called *G-primitive.* ∎

We are only concerned with finite-dimensional modules. Thus, weakly $G_{\overline{0}}$-primitive vectors is in fact $G_{\overline{0}}$-primitive and integral dominant. A *cyclic module* is an indecomposable module generated by a weakly primitive vector. A weakly primitive vector $v$ will determine a cyclic submodule $\mathbf{U}(G)v$ and a composition factor. An important construct in classifying composition factors is the atypicality matrix $A(\Lambda)$.[2,15,16] First, introduce the shorthand notation:

$$\beta_{bc} = \epsilon_b - \delta_c = \alpha_{\overline{m-b+1},c-1}, \ 1 \le b \le m+1, 1 \le c \le n+1.$$

*Definition 3.5.* The atypicality matrix $A(\Lambda)$ is the $(m+1)\times(n+1)$ matrix with $(b,c)$-entry $A(\Lambda)_{bc}$ $=\langle\Lambda+\rho|\beta_{bc}\rangle=\sum_{k=\overline{m-b+1}}^{0} a_k - \sum_{k=1}^{c-1} a_k + m - b - c + 2$. For example, for $G=sl(4/5)$, $\Lambda=[100;0;1000]$,

$$A(\Lambda) = \begin{pmatrix} 4 & 2 & 1 & 0 & \overline{1} \\ 2 & 0 & \overline{1} & \overline{2} & \overline{3} \\ 1 & \overline{1} & \overline{2} & \overline{3} & \overline{4} \\ 0 & \overline{2} & \overline{3} & \overline{4} & \overline{5} \end{pmatrix}.$$ ∎

Inspection of $A(\Lambda)$ tells immediately whether or not $\Lambda$ is atypical and which are the atypical roots since they correspond to zero entries of $A(\Lambda)$. In above, $\Lambda$ is 3-fold atypical with atypical roots $\beta_{41}$, $\beta_{22}$, $\beta_{14}$. The properties of $A(\Lambda)$ have been studied in detail in *Ref.* 2. We summarize some here.

*Lemma 3.6.* (i) Let $\Lambda = [a_{\overline{m}}, \cdots, a_{\overline{1}}; a_0; a_1, \cdots, a_n]$; then

$$\begin{aligned} A(\Lambda)_{bc} - A(\Lambda)_{b+1,c} &= a_{\overline{m-b+1}} + 1, \ 1 \le b \le m, 1 \le c \le n+1, \\ A(\Lambda)_{m+1,1} &= a_0, \\ A(\Lambda)_{bc} - A(\Lambda)_{b,c+1} &= a_c + 1, \ 1 \le b \le m+1, 1 \le c \le n. \end{aligned} \tag{3.1a}$$

(ii) An atypicality matrix $A(\Lambda)$ satisfies $A(\Lambda)_{bc} + A(\Lambda)_{de} = A(\Lambda)_{be} + A(\Lambda)_{dc}$. *Vice versa*, any $(m+1)\times(n+1)$ matrix satisfying this condition for all pairs $(b,c)$, $(d,e)$ with $1 \le b,d \le m+1$ and $1 \le c,e \le n+1$ is the atypicality matrix of a unique element $\Lambda \in H^*$.

(iii) $\Lambda$ is dominant $\Leftrightarrow$

$$\begin{array}{ll} A(\Lambda)_{bc} - A(\Lambda)_{b+1,c} - 1 \geq 0, & 1 \leq b \leq m, 1 \leq c \leq n+1, \\ A(\Lambda)_{bc} - A(\Lambda)_{b,c+1} - 1 \geq 0, & 1 \leq b \leq m+1, 1 \leq c \leq n. \end{array} \tag{3.1b}$$

Moreover, $\Lambda$ is integral if the expressions on the *l.h.s.* of (3.1b) are all integers. ∎

For atypical modules, the highest weight $\Lambda$ must be integral dominant and $a_0$ is an integer since at least one of the entries of $A(\Lambda)$ is zero. Lemma 3.6 implies that the zeros of $A(\Lambda)$ lie in distinct rows and columns, and that one zero lies to the right of another $\Leftrightarrow$ it lies above it. Thus the atypical roots are commensurate with respect to ordering (2.3). If $\Lambda$ is $r$-fold atypical, we label the atypical roots $\gamma_1 < \cdots < \gamma_r$. It follows that if $1 \leq s, t \leq r$ and $x_{st}$ is the entry in $A(\Lambda)$ at the intersection of the column containing the $\gamma_s$ zero with the row containing the $\gamma_t$ zero, then $x_{st} \in \mathbb{Z}_+ \backslash \{0\}$ for $s < t$ and $x_{ts} = -x_{st}$. Therefore $A(\Lambda)$ has the form:

$$A(\Lambda) = \begin{pmatrix} & \vdots & & \vdots & & \vdots & & \vdots & \\ \cdots & x_{1r} & \cdots & x_{2r} & \cdots & x_{3r} & \cdots & 0 & \cdots \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ \cdots & x_{13} & \cdots & x_{23} & \cdots & 0 & \cdots & \overline{x_{3r}} & \cdots \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ \cdots & x_{12} & \cdots & 0 & \cdots & \overline{x_{23}} & \cdots & \overline{x_{2r}} & \cdots \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ \cdots & 0 & \cdots & \overline{x_{12}} & \cdots & \overline{x_{13}} & \cdots & \overline{x_{1r}} & \cdots \end{pmatrix}. \tag{3.2}$$

Denote $h_{st}$ the hook length between the zeros corresponding to $\gamma_s, \gamma_t$, i.e., the number of steps to go from the $\gamma_s$ zero via $x_{st}$ to the $\gamma_t$ zero with the zeros themselves included in the count. An important concept in the classification of composition factors is the following.[2]

*Definition 3.7.* Let $\Lambda$ be $r$-fold atypical with atypical roots $\{\gamma_1, \cdots, \gamma_r\}$. For $1 \leq s < t \leq r$:

(i) $\gamma_s, \gamma_t$ are *normally related* $(n)$ $\quad\Leftrightarrow\quad x_{st} > h_{st} - 1$;
(ii) $\gamma_s, \gamma_t$ are *quasi-critically related* $(q)$ $\quad\Leftrightarrow\quad x_{st} = h_{st} - 1$;
(iii) $\gamma_s, \gamma_t$ are *critically related* $(c)$ $\quad\Leftrightarrow\quad x_{st} < h_{st} - 1$. ∎

It is straightforward to show that the $q$-relation is transitive, i.e., if $\gamma_s, \gamma_t$ are $q$-related and $\gamma_t, \gamma_u$ are $q$-related, then $\gamma_s, \gamma_u$ are $q$-related.

*Definition 3.8.* The *nqc-type* (*atypicality type*) of an $r$-fold atypical $\Lambda$ is a triangular array

$$nqc(\Lambda) = \begin{array}{lllllll} s_{1r} & \cdots & s_{sr} & \cdots & s_{tr} & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \iddots & \\ s_{1t} & \cdots & s_{st} & \cdots & 0 & & \\ \vdots & & \vdots & \iddots & & & \\ s_{1s} & \cdots & 0 & & & & \\ \vdots & \iddots & & & & & \\ 0 & & & & & & \end{array} , \qquad s_{st} \in \{n, q, c\},$$

where the zeros correspond to $\{\gamma_1, \cdots, \gamma_r\}$ and $s_{st} = n, q, c \Leftrightarrow \gamma_s, \gamma_t$ are $n$-, $q$-, $c$-related. ∎

It was conjectured[2] that the number and nature of composition factors of $\overline{V}(\Lambda)$ depends only on *nqc*-type of $\Lambda$; if a weight $\Lambda$ of $sl(m+1/n+1)$ and a weight $\Lambda'$ of $sl(m'+1/n'+1)$ have the same *nqc*-type, then there is a 1 - 1 correspondence between the composition factors of $\overline{V}(\Lambda)$ and $\overline{V}(\Lambda')$. More precisely it was conjectured that the composition factors of $\overline{V}(\Lambda)$ are in

1 - 1 correspondence with certain codes $\Sigma^c$ which are determined from $nqc(\Lambda)$, and which in turn determine the highest weights $\Sigma$ of the corresponding composition factors $V(\Sigma)$.

*Definition 3.9.* A *code* $\Sigma^c$ for $\Lambda$ is an array of length $r$, each element of the array consisting of a non-empty column of increasing labels taken from $\{0, \cdots, r\}$. The 1st element of a column is called the *top label*. $\Sigma^c$ must satisfy the rules:

(i) The top label of column $s$ can be $0$, $s$ or $a$ with $s<a$; the 1st case can occur only if column $s$ is zero, while the last case can occur only if $nqc(\Lambda)_{st}=q$ with $a$ the top label of column $t$.

(ii) Let $s < t, nqc(\Lambda)_{st} = \cdots = nqc(\Lambda)_{t-1,t} = c$. If the top label of column $t$ is $a : t \leq a$, then $a$ must appear somewhere below the top entry of column $s$.

(iii) If $s$ appears in any column then the only labels which can appear below $s$ in the same column are those $t : s < t$, for which $t$ is the top label of column $t$ and $nqc(\Lambda)_{st} = c$.

(iv) If the label $s$ appears in more than one column and $t$ appears immediately below $s$ in one such column, then it must do so in all columns containing $s$.

(v) Let $s < t < u$, $nqc(\Lambda)_{st} = q, nqc(\Lambda)_{tu} = q$ (so, $nqc(\Lambda)_{su} = q$). If the top label of column $s$ is the same as that of column $u$ and it is non-zero then the top label of column $t$ is not 0.

(vi) Let $s < t < u < v$ with top labels $a, b, a, b$ respectively, $a \neq 0 \neq b$. If $a < b$ then columns $s$ and $u$ must contain $b$; if $a > b$ then columns $t$ and $v$ must contain $a$. ∎

As an example, consider $\Lambda = [00020; 0; 0210]$ for $sl(6/5)$. A straightforward computation gives

$$A(\Lambda) = \begin{pmatrix} 7 & 6 & 3 & 1 & 0 \\ 6 & 5 & 2 & 0 & \overline{1} \\ 5 & 4 & 1 & \overline{1} & \overline{2} \\ 4 & 3 & 0 & \overline{2} & \overline{3} \\ 1 & 0 & \overline{3} & \overline{5} & \overline{6} \\ 0 & \overline{1} & \overline{4} & \overline{6} & \overline{7} \end{pmatrix} , \qquad nqc(\Lambda) = \begin{matrix} c & c & c & c & 0 \\ c & q & c & 0 & \\ q & n & 0 & & \\ c & 0 & & & \\ 0 & & & & \end{matrix} . \tag{3.3}$$

$\Lambda$ is 5-fold atypical with atypical roots $\gamma_1 = \beta_{61}, \gamma_2 = \beta_{52}, \gamma_3 = \beta_{43}, \gamma_4 = \beta_{24}, \gamma_5 = \beta_{15}$. Using the rules in Definition 3.9, we find the following 15 codes $\Sigma^c$:

$$\begin{gathered} \begin{smallmatrix} 0&0&0&0&0 \end{smallmatrix} \quad \begin{smallmatrix} 1&0&0&0&0 \end{smallmatrix} \quad \begin{smallmatrix} 1&2&0&0&0 \\ 2&&&& \end{smallmatrix} \quad \begin{smallmatrix} 0&0&3&0&0 \end{smallmatrix} \quad \begin{smallmatrix} 1&0&3&0&0 \end{smallmatrix} \quad \begin{smallmatrix} 1&2&3&0&0 \\ 2&&&& \end{smallmatrix} \quad \begin{smallmatrix} 3&0&3&0&0 \end{smallmatrix} \quad \begin{smallmatrix} 0&0&3&4&0 \\ &&4&& \end{smallmatrix} \\ \begin{smallmatrix} 1&0&3&4&0 \\ &&4&& \end{smallmatrix} \quad \begin{smallmatrix} 1&2&3&4&0 \\ 2&&4&& \end{smallmatrix} \quad \begin{smallmatrix} 1&4&3&4&0 \\ 4&&4&& \end{smallmatrix} \quad \begin{smallmatrix} 3&4&3&4&0 \\ 4&&4&& \end{smallmatrix} \quad \begin{smallmatrix} 1&2&3&4&5 \\ 2&5&4&5& \\ 5&&5&& \end{smallmatrix} \quad \begin{smallmatrix} 1&4&3&4&5 \\ 4&5&4&5& \\ 5&&5&& \end{smallmatrix} \quad \begin{smallmatrix} 3&4&3&4&5 \\ 4&5&4&5& \\ 5&&5&& \end{smallmatrix} . \end{gathered} \tag{3.4}$$

For $1 \leq s \leq r$, the $s$-th column of a code corresponds to the $s$-th atypical root $\gamma_s$. From definition, we see that if $\gamma_s, \gamma_t$ are $q$-related and the top entry $a$ of column $s$ is non-zero, then $a$ may also be the top entry of column $t$. In such a case, we say that $\gamma_s, \gamma_t$ are *linked*. In the example, $\gamma_1, \gamma_3$ are $q$-related, and they are linked in code (30300), whereas in code (10300), they are not. Thus, where $\gamma_s, \gamma_t$ are $q$-related, there will be codes in which they are linked, and codes in which they are not. This leads the following definition.

*Definition 3.10.* A code $\Sigma^c$ is a *linked code* if there exist $\gamma_s, \gamma_t$ which are linked, i.e., columns $s$ and $t$ have the same non-zero top entry. Otherwise, it is called an *unlinked code*. ∎

It follows from rule (i) that if $nqc(\Lambda)$ contains no $q$, then all codes are unlinked. Next, we see from rule (ii) that if, for $s < t$, $\gamma_s, \cdots, \gamma_{t-1}$ are $c$-related to $\gamma_t$ and if the top label $a$ of column $t$ of a code is non-zero, then $a$ must appear somewhere below the top entry of the $s$-th column, and

so also of the $(s{+}1)$-th,$\cdots$,$(t$-$1)$-th column, of that code; we say $\gamma_t$ *wraps* $\gamma_s$. Unlike links, wraps must be made. In the example, $\gamma_1, \gamma_2$ are $c$-related, and each code in which the 2nd column is non-zero, e.g., $\begin{smallmatrix}1&2&0&0&0\\2&&&&\end{smallmatrix}$, $\begin{smallmatrix}1&4&3&4&0\\4&&4&&\end{smallmatrix}$, the top label of the 2nd column occurs below the top entry in the 1st column; i.e., $\gamma_2$ wraps $\gamma_1$. Similarly, $\gamma_1, \cdots, \gamma_4$ are $c$-related to $\gamma_5$, and in each code with non-zero top label in the 5th column, that label occurs below the top entry in each of the first 4 columns, as in $\begin{smallmatrix}3&4&3&4&5\\4&5&4&5&\\5&&5&&\end{smallmatrix}$. It may happen, on the other hand, that for $s < t, \gamma_s, \gamma_t$ are $c$-related but $\gamma_u, \gamma_t$ are not $c$-related for some $u$, $s < u < t$, as in (3.3) $\gamma_1, \gamma_4$ are $c$-related but $\gamma_2, \gamma_4$ are $q$-related. Correspondingly, in code $\begin{smallmatrix}1&2&3&4&0\\2&&4&&\end{smallmatrix}$, $\gamma_4$ does not wrap $\gamma_1$. However, in code $\begin{smallmatrix}1&4&3&4&0\\4&&4&&\end{smallmatrix}$, $\gamma_4$ does appear to wrap $\gamma_1$. This is because in this code, as opposed to the previous one, $\gamma_4$ is linked to $\gamma_2$, so the top label of the 2nd column is the same (i.e., 4) as that of the 4th column, and therefore, since $\gamma_2$ must wrap $\gamma_1$ in all codes in which the 2nd column is non-zero, this entry must appear below the top entry in the 1st column. Thus $\gamma_4$ wraps $\gamma_1$ only because of the presence of an intermediate link; we shall use the term *link wrap* rather than wrap to describe this. From the discussion we see that, in general, the presence of a $q$ rather than an $n$ in $nqc(\Lambda)$ increases, whereas the presence of a $c$ rather than an $n$ decreases, the number of codes for $\Lambda$. Thus, for $r = 2$, $\Lambda$ has $3, 4, 5$ codes $\Leftrightarrow nqc(\Lambda) = \begin{smallmatrix}c&0\\0&\end{smallmatrix}, \begin{smallmatrix}n&0\\0&\end{smallmatrix}, \begin{smallmatrix}q&0\\0&\end{smallmatrix}$.

*Definition 3.11.* In a code $\Sigma^c$ for $\Lambda$, we say that $\gamma_s$ is *connected* to $\gamma_t$ and write $\gamma_s \underset{\Sigma}{\sim} \gamma_t$ if the $s$-th and $t$-th columns of $\Sigma^c$ contain a common non-zero entry. ∎

Thus for the code $\Sigma^c = \begin{smallmatrix}1&2&3\\3&3&\end{smallmatrix}$, $\gamma_1 \underset{\Sigma}{\sim} \gamma_2 \underset{\Sigma}{\sim} \gamma_3$, whereas for the code $\Sigma^c = 1\,2\,0$, $\gamma_1 \underset{\Sigma}{\not\sim} \gamma_2$. Clearly, $\underset{\Sigma}{\sim}$ is an equivalent relation on $\{\gamma_s \,|\, \text{the } s\text{-th column of } \Sigma \text{ is non-zero}\}$.

*Definition 3.12.* A code $\Sigma^c$ is called *indecomposable* if $\{\gamma_s \,|\, \text{the } s\text{-th column of } \Sigma^c \text{ is non-zero}\}$ is an equivalent class for the relation $\underset{\Sigma}{\sim}$; otherwise $\Sigma^c$ is called *decomposable*. ∎

For example, $\begin{smallmatrix}1&2&3&0\\3&3&&\end{smallmatrix}$ is indecomposable, whereas $\begin{smallmatrix}1&2&3&4\\2&&4&\end{smallmatrix}$ is decomposable. If $\Sigma^c$ is decomposable, then we can write $\Sigma^c = \Sigma^c_1 \Sigma^c_2 \cdots \Sigma^c_s \cdots$, where each $0 \cdots 0 \Sigma^c_s 0 \cdots 0$ (with 0's in the appropriate positions) is indecomposable. For instance, $\Sigma^c = \begin{smallmatrix}1&2&3&4\\2&&4&\end{smallmatrix}$ can be written as $\Sigma^c = \Sigma^c_1 \Sigma^c_2$ where $\Sigma^c_1 = \begin{smallmatrix}1&2\\2&\end{smallmatrix}$, $\Sigma^c_2 = \begin{smallmatrix}3&4\\4&\end{smallmatrix}$, with $\begin{smallmatrix}1&2&0&0\\2&&&\end{smallmatrix}$ and $\begin{smallmatrix}0&0&3&4\\&&4&\end{smallmatrix}$ being themselves indecomposable unlinked codes for $\Lambda$. Without confusion we will simply denote $0 \cdots 0 \Sigma^c_s 0 \cdots 0$ by $\Sigma^c_s$.

## IV. SOUTH WEST CHAINS OF $A(\Lambda)$

To obtain the highest weights of those composition factors of $\overline{V}(\Lambda)$ corresponding to unlinked codes, we need to construct south west chains.[2] Denote by

$$D = \{(b, c) | 1 \leq b \leq m + 1, 1 \leq c \leq n + 1\}, \qquad \Gamma_\Lambda = \{(b, c) | A(\Lambda)_{bc} = 0\}. \tag{4.1}$$

the set of, respectively, positions, positions of zeros, of $A(\Lambda)$ and define $\widehat{K} = \{\beta_{bc} \,|\, (b, c) \in K\}$ for any subset $K$ of $D$. In particular, $\widehat{D} = \Delta_1^+$ and $\widehat{\Gamma}_\Lambda = \{\gamma_1, \ldots, \gamma_r\}$.

*Definition 4.1.* (i) For $1 \le s \le r$, let $(b_s,c_s) \in \Gamma_\Lambda$ be the position corresponding to $\gamma_s$. The *extended west chain* $W^e_\Lambda(s)$ emanating from $(b_s,c_s)$ is a sequence of positions in $D$ starting at $(b_s,c_s)$ and extending in a westerly or south-westerly direction until it reaches the 1st column or it cannot extend further without leaving $A(\Lambda)$ by passing below its bottom row. For all $t$ with $1 \le t \le c_s$, $W^e_\Lambda(s)$ has exactly one element in the $t$-th column provided that the row of this element lies within $A(\Lambda)$. For $1 \le t \le c_s - 1$, the row of the position in the $t$-th column is $a_t$ rows below the row of the position in the $(t+1)$-th column, where $a_t$ is a Dynkin label of $\Lambda$; if this is not possible, i.e., if this row would be the $M$-th row where $M > m+1$, then $W^e_\Lambda(s)$ ends in the $t$-th column, i.e., has no position to the left of the $t$-th column. Thus $W^e_\Lambda(s)$ is the set

$$W^e_\Lambda(s) = D \cap \{(b,c) | 1 \le c \le c_s, b = b_s + \sum_{t=c}^{c_s-1} a_t\}. \tag{4.2a}$$

(ii) Similarly, the *extended south chain* $S^e_\Lambda(s)$ emanating from $(b_s, c_s)$ is the set

$$S^e_\Lambda(s) = D \cap \{(b,c) | b_s \le b \le m+1, c = c_s - \sum_{t=b_s}^{b-1} a_{\overline{m-t+1}}\}. \tag{4.2b}$$

(iii) The *extended south west chain* emanating from $(b_s,c_s)$ is $SW^e_\Lambda(s) = W^e_\Lambda(s) \cup S^e_\Lambda(s)$. ∎

*Definition 4.2.* (i) For $1 \le s \le r$, the *south west chain* $SW_\Lambda(s)$ emanating from $(b_s, c_s)$ consists of all positions of $SW^e_\Lambda(s)$ which are above and to the right of **all** points of intersection of the chains $W^e_\Lambda(s)$ and $S^e_\Lambda(s)$ of $A(\Lambda)$, with the additional requirement that if $S^e_\Lambda(s)$ starts off at $(b_s, c_s)$ by immediately going above $W^e_\Lambda(s)$, then $SW_\Lambda(s)$ consists solely of $(b_s, c_s)$. The *west, south subchain* of $SW_\Lambda(s)$ are, respectively, $W_\Lambda(s) = W^e_\Lambda(s) \cap SW_\Lambda(s)$, $S_\Lambda(s) = S^e_\Lambda(s) \cap SW_\Lambda(s)$.

(ii) $SW_\Lambda = \cup_{s=1}^r SW_\Lambda(s)$ is called the set of all south west chains. ∎

The construction of chains is facilitated by placing the Dynkin labels $a_{\overline{m}}, ..., a_{\overline{1}}$ to the left of the 1st column, and in between the rows, of $A(\Lambda)$, likewise, $a_1, ..., a_n$ are placed below the bottom row, and in between the columns, of $A(\Lambda)$. We illustrate chains with 3 examples of doubly atypical $\Lambda$ for $sl(5/6)$. In each case we first give $SW^e_\Lambda(s)$, denoting $W^e_\Lambda(s)$ by broken lines, $S^e_\Lambda(s)$ by unbroken lines, and then give $SW_\Lambda(s)$, denoting the chains by arrows.

*Example 4.3.* $\Lambda = [1\,1\,1\,1; \overline{1}; 1\,0\,0\,1\,0]$.

$$A(\Lambda) = \begin{array}{c} \\ 1 \\ 1 \\ 1 \\ 1 \\ \\ \end{array} \begin{pmatrix} 7 & 5 & 4 & 3 & 1 \cdots & 0 \\ 5 & 3 \cdots & 2 \cdots & 1 & \overline{1} & \overline{2} \\ 3 & 1 \cdots & 0 & \overline{1} & \overline{3} & \overline{4} \\ 1 & \overline{1} & \overline{2} & \overline{3} & \overline{5} & \overline{6} \\ \overline{1} & \overline{3} & \overline{4} & \overline{5} & \overline{7} & \overline{8} \end{pmatrix}_{\;1\;\;0\;\;0\;\;1\;\;0} = \begin{array}{c} \\ 1 \\ 1 \\ 1 \\ 1 \\ \\ \end{array} \begin{pmatrix} 7 & 5 & 4 & 3 & 1 \leftarrow & 0 \\ 5 & 3 \leftarrow & 2 \leftarrow & 1 & \overline{1} & \overline{2} \\ 3 & 1 \leftarrow & 0 & \overline{1} & \overline{3} & \overline{4} \\ 1 & \overline{1} & \overline{2} & \overline{3} & \overline{5} & \overline{6} \\ \overline{1} & \overline{3} & \overline{4} & \overline{5} & \overline{7} & \overline{8} \end{pmatrix}_{\;1\;\;0\;\;0\;\;1\;\;0}.$$

$SW^e_\Lambda(1) = SW_\Lambda(1) = \{(3,2), (3,3), (4,1), (4,2), (5,1)\}$; $SW^e_\Lambda(2) = SW_\Lambda(2) = \{(1,5), (1,6), (2,2), (2,3), (2,4), (2,5), (3,1), (3,4), (4,3), (5,2)\}$. Here $\gamma_1, \gamma_2$ are $c$-related. ∎

*Example 4.4.* $\Lambda = [2\ 3\ 0\ 2; \overline{1}; 0\ 1\ 1\ 2\ 1]$.

$$A(\Lambda) = \begin{array}{c} 2 \\ 3 \\ 0 \\ 2 \end{array} \begin{array}{c} \begin{pmatrix} 10 & 9 & 7 & 5 & 2 & 0 \\ 7 & 6 & 4 & 2 & \overline{1} & \overline{3} \\ 3 & 2 & 0 & \overline{2} & \overline{5} & \overline{7} \\ 2 & 1 & \overline{1} & \overline{3} & \overline{6} & \overline{8} \\ \overline{1} & \overline{2} & \overline{4} & \overline{6} & \overline{9} & \overline{11} \end{pmatrix} \\ 0 \quad 1 \quad 1 \quad 2 \quad 1 \end{array} = \begin{array}{c} 2 \\ 3 \\ 0 \\ 2 \end{array} \begin{array}{c} \begin{pmatrix} 10 & 9 & 7 & 5 & 2 & 0 \\ 7 & 6 & 4 & 2 & \overline{1} & \overline{3} \\ 3 & 2 & 0 & \overline{2} & \overline{5} & \overline{7} \\ 2 & 1 & \overline{1} & \overline{3} & \overline{6} & \overline{8} \\ \overline{1} & \overline{2} & \overline{4} & \overline{6} & \overline{9} & \overline{11} \end{pmatrix} \\ 0 \quad 1 \quad 1 \quad 2 \quad 1 \end{array}.$$

$SW^e_\Lambda(1) = SW_\Lambda(1) = \{(3,3),\ (4,1),\ (4,2),\ (4,3),\ (5,1)\}$. However, $SW^e_\Lambda(2) = \{(1,6),\ (2,4),\ (2,5),\ (3,1),\ (4,1),\ (4,4),\ (5,3)\}$, while $SW_\Lambda(2) = \{(1,6)\}$. In this case $\gamma_1, \gamma_2$ are $n$-related. ∎

*Example 4.5.* $\Lambda = [1\ 2\ 1\ 1; \overline{1}; 1\ 0\ 0\ 0\ 2]$.

$$A(\Lambda) = \begin{array}{c} 1 \\ 2 \\ 1 \\ 1 \end{array} \begin{array}{c} \begin{pmatrix} 8 & 6 & 5 & 4 & 3 & 0 \\ 6 & 4 & 3 & 2 & 1 & \overline{2} \\ 3 & 1 & 0 & \overline{1} & \overline{2} & \overline{5} \\ 1 & \overline{1} & \overline{2} & \overline{3} & \overline{4} & \overline{7} \\ \overline{1} & \overline{3} & \overline{4} & \overline{5} & \overline{6} & \overline{9} \end{pmatrix} \\ 1 \quad 0 \quad 0 \quad 0 \quad 2 \end{array} = \begin{array}{c} 1 \\ 2 \\ 1 \\ 1 \end{array} \begin{array}{c} \begin{pmatrix} 8 & 6 & 5 & 4 & 3 & 0 \\ 6 & 4 & 3 & 2 & 1 & \overline{2} \\ 3 & 1 & 0 & \overline{1} & \overline{2} & \overline{5} \\ 1 & \overline{1} & \overline{2} & \overline{3} & \overline{4} & \overline{7} \\ \overline{1} & \overline{3} & \overline{4} & \overline{5} & \overline{6} & \overline{9} \end{pmatrix} \\ 1 \quad 0 \quad 0 \quad 0 \quad 2 \end{array}.$$

$SW^e_\Lambda(1) = SW_\Lambda(1) = \{(3,2),\ (3,3),\ (4,1),\ (4,2),\ (5,1)\}$. However, $SW^e_\Lambda(2) = \{(1,6),\ (2,5),\ (3,2),\ (3,3),\ (3,4),\ (3,5),\ (4,1),\ (4,2),\ (5,1)\}$ while $SW_\Lambda(2) = \{(1,6)\}$. Because $S^e_\Lambda(2), W^e_\Lambda(2)$ meet at the zero corresponding to $\gamma_1, SW^e_\Lambda(1) \subset SW^e_\Lambda(2)$, and $\gamma_1, \gamma_2$ are $q$-related. ∎

Although we shall not give details here, it is easy to check that for doubly atypical $\Lambda = [4020; \overline{2}; 00120], \gamma_1, \gamma_2$ are $n$-related and $W^e_\Lambda(2)$ starts off above $S^e_\Lambda(2)$ but crosses it at a point which is above the zero corresponding to $\gamma_1$. Also, $\Lambda = [1220; \overline{2}; 00110]$ is doubly atypical with $\gamma_1, \gamma_2$ being $q$-related; in this case, $W^e_\Lambda(2)$ starts off above $S^e_\Lambda(2)$ but meets it at the position of the zero corresponding to $\gamma_1$; the position of this zero is therefore not an element of $SW_\Lambda(2)$. These examples illustrate some properties of chains we are now going to state.

*Lemma 4.6.* Let $s < t$. Then $\gamma_s, \gamma_t$ are $q$-related $\Leftrightarrow W^e_\Lambda(t), S^e_\Lambda(t)$ both contain $(b_s, c_s)$.

*Proof.* Since $A(\Lambda)_{b_s,c_s} = A(\Lambda)_{b_t,c_t} = 0$, we have $x_{st} = A(\Lambda)_{b_t,c_s} = c_t - c_s + \sum_{i=c_s}^{c_t-1} a_i$. Also $h_{st} = b_s - b_t + c_t - c_s + 1$. Therefore, $\gamma_s, \gamma_t$ are $q$-related $\Leftrightarrow x_{st} = h_{st} - 1 \Leftrightarrow b_s = b_t + \sum_{i=c_s}^{c_t-1} a_i$ $\Leftrightarrow (b_s, c_s) \in W^e_\Lambda(t)$. Similarly, if $\gamma_s, \gamma_t$ are $q$-related, $(b_s, c_s) \in S^e_\Lambda(t)$. ∎

*Lemma 4.7.* If $s < t, \gamma_s, \gamma_t$ are $n$-related, then $W^e_\Lambda(t)$ meets column $c_s$ below $(b_s, c_s)$, and $S^e_\Lambda(t)$ meets row $b_s$ to the left of $(b_s, c_s)$.

*Proof.* Since $\gamma_s, \gamma_t$ are $n$-related, $x_{st} > h_{st} - 1$, hence $b_s < b_t + \sum_{i=c_s}^{c_t-1} a_i$. Hence $W^e_\Lambda(t)$ meets column $c_s$ below $(b_s, c_s)$. The proof of the other result is analogous. ∎

*Lemma 4.8.* If $s < t, \gamma_s, \gamma_t$ are $q$- or $n$-related, then $SW_\Lambda(t)$ does not extend as far to the left

or downwards as $(b_s,c_s)$, i.e., $W_\Lambda(t)$ ends to the right of column $c_s$, $S_\Lambda(t)$ ends above row $b_s$.

*Proof.* If $\gamma_s, \gamma_t$ are $q$-related, by Lemma 4.6 $W^e_\Lambda(t), S^e_\Lambda(t)$ meet at $(b_s, c_s)$. Otherwise, by Lemma 4.7 $W^e_\Lambda(t)$,$S^e_\Lambda(t)$ must cross to the right of column $c_s$ and above row $b_s$. In either case, by Definition 4.2 $SW_\Lambda(t)$ does not extend as far to the left or downwards as $(b_s,c_s)$. ∎

*Lemma 4.9.* Let $1 \leq t \leq r$ and $b_t \leq d \leq m+1$, $1 \leq e \leq c_t$ and $i = A(\Lambda)_{de} \in \mathbb{Z}$. If $(d,e) \in W^e_\Lambda(t)$ then $(d, e+i) \in S^e_\Lambda(t)$. If $(d,e) \in S^e_\Lambda(t)$ then $(d+i, e) \in W^e_\Lambda(t)$.

*Proof.* If $(d,e) \in W^e_\Lambda(t)$, then $d = b_t + \sum_{j=e}^{c_t-1} a_j$. We have $0 = A(\Lambda)_{b_t,c_t} = A(\Lambda)_{de} + \sum_{j=m-d+2}^{m-b_t+1} a_j - \sum_{j=e}^{c_t-1} a_j + d - b_t - c_t + e$. So $e + i = e + A(\Lambda)_{de} = c_t - \sum_{j=m-d+2}^{m-b_t+1} a_j$, by (4.2b), $(d, e+i) \in S^e_\Lambda(t)$. Similarly, if $(d,e) \in S^e_\Lambda(t)$ then $(d+i, e) \in W^e_\Lambda(t)$. ∎

*Lemma 4.10.* In Lemma 4.9, if $(d,e) \in W_\Lambda(t)$, then $i \geq 0$ and $(d, e+i) \in S_\Lambda(t)$. If $(d,e) \in S_\Lambda(t)$, then $i = -j \leq 0$ and $(d-j, e) \in W_\Lambda(t)$.

*Proof.* Let's say $(d,e) \in W_\Lambda(t)$. By Lemma 4.9 $(d,e{+}i) \in S^e_\Lambda(t)$. Since $(d,e) \in W_\Lambda(t)$, by Definition 4.2, $S^e_\Lambda(t)$,$W^e_\Lambda(t)$ must have not crossed each other above or to the right of $(d,e)$ and so $(d,e{+}i)$ must be to the right of $(d,e)$. Hence $i{\geq}0$ and $(d,e{+}i){\in}S_\Lambda(t)$. ∎

In Example 4.3, $(2,2) \in W_\Lambda(2)$, $A(\Lambda)_{22} = 3$; one has $(5,2) \in S_\Lambda(2)$. Also, $(5,1) \in S_\Lambda(1)$, $A(\Lambda)_{51} = -1$; one has $(4,1) \in W_\Lambda(1)$. Finally, $(3,3) \in W_\Lambda(1) \cap S_\Lambda(1)$, $A(\Lambda)_{33} = 0$. Obviously $W_\Lambda(1)$ is zero rows above $(3,3)$ and $S_\Lambda(1)$ is zero rows to the right of $(3,3)$.

*Lemma 4.11.* (i) Let $(d,e) \in W_\Lambda(t), 0 \leq k \leq m+1-d$. If $A(\Lambda)_{d+k,e} = j$, $j+k \geq 0$, then $S_\Lambda(t)$ meets the $(d+k)$-th row $(j+k)$ columns to the right of $(d+k, e)$.

(ii) Let $(d,e) \in S_\Lambda(t), 0 \leq k \leq e-1$. If $A(\Lambda)_{d,e-k} = -j$, $j+k \geq 0$, then $W_\Lambda(t)$ meets the $(e-k)$-th column $(j+k)$ rows above $(d, e-k)$.

*Proof.* Let's prove (i) as (ii) is similar. Let $0 \leq k' \leq k$, $A(\Lambda)_{d+k',e} = j'$, then $j' \geq j + k - k'$ and so $j' + k' \geq j + k \geq 0$. The condition of the lemma is satisfied for $k'$. Let $A(\Lambda)_{de} = i$; by Lemma 4.10, $i \geq 0$, $(d, e+i) \in S_\Lambda(t)$. By (4.2b), $(d+k', e+i-\sum_{\ell=m-d+1}^{m-d-k'+2} a_\ell) \in S^e_\Lambda(t)$. Now $A(\Lambda)_{d+k',e}{=}A(\Lambda)_{d,e}{-}k'{-}\sum_{\ell=m-d+1}^{m-d-k'+2} a_\ell$, so $j'{=}i{-}k'{-}\sum_{\ell=m-d+1}^{m-d-k'+2} a_\ell$ and $(d{+}k', e{+}j'{+}k'){\in}S^e_\Lambda(t)$. Hence, since $j'+k' \geq 0$ for all $k'$, $(d+k', e+j'+k')$ is to the right of column $e$, whereas $W^e_\Lambda(t)$ is to the left of column $e$ after having passed at $(d,e)$. Thus $(d+k', e+j'+k') \in S_\Lambda(t)$. So $(d+k, e+j+k) \in S_\Lambda(t)$ and the result follows. ∎

*Definition 4.12.* Suppose $W_\Lambda(t), S_\Lambda(t)$ end at $(d_t, e_t), (d'_t, e'_t)$, respectively, $d'_t \geq d_t, e'_t \geq e_t$. Define $D(t)$ to be the region of $D$ within or on the boundary consisting of $SW_\Lambda(t)$, the vertical line joining $(d'_t, e_t)$ to $(d_t, e_t)$ and the horizontal line joining $(d'_t, e_t)$ to $(d'_t, e'_t)$. ∎

We will prove that every element of $D(t)$ lies in some $SW_\Lambda(s)$ such that $s{\leq}t$, $\gamma_s, \gamma_t$ are $c$-related.

*Lemma 4.13.* (i) If $A(\Lambda)_{bc} > 0$ and $A(\Lambda)_{b-a_c,c+1} \leq 0$ then $A(\Lambda)_{bc} = 1$ and $A(\Lambda)_{b-a_c,c+1} = 0$.

(ii) If $A(\Lambda)_{bc} < 0$ and $A(\Lambda)_{b-1,c+a_{\overline{m-b+2}}} \geq 0$ then $A(\Lambda)_{bc} = -1$ and $A(\Lambda)_{b-1,c+a_{\overline{m-b+2}}} = 0$.

*Proof.* The proof of (ii) is analogous to that of (i). For (i), using (3.1a),

$$A(\Lambda)_{b-a_c,c+1} = A(\Lambda)_{bc} - (a_c+1) + \sum_{\ell=\overline{m-b+2}}^{\overline{m-b+1+a_c}} a_\ell + a_c = A(\Lambda)_{bc} - 1 + \sum_{\ell=\overline{m-b+2}}^{\overline{m-b+1+a_c}} a_\ell \leq 0.$$

The only solution is $a_{\overline{m-b+2}}{=}\cdots{=}a_{\overline{m-b+1+a_c}}{=}0$, $A(\Lambda)_{bc}{=}1$, $A(\Lambda)_{b-a_c,c+1}{=}0$. ∎

*Lemma 4.14.* Let $(d,e) \in D(t)$.

(i) If $A(\Lambda)_{de} \geq 0$, then there exists $s$, $1 \leq s \leq t$ such that $(b_s,c_s) \in D(t)$, $(d,e) \in W_\Lambda(s)$;

(ii) If $A(\Lambda)_{de} \leq 0$, then there exists $s$, $1 \leq s \leq t$ such that $(b_s,c_s) \in D(t)$, $(d,e) \in S_\Lambda(s)$;

(iii) In both of these cases, $\gamma_s, \gamma_t$ are $c$-related.

*Proof.* (i) Suppose $A(\Lambda)_{de} \geq 0$ and let $k$ be such that $(d-k,e) \in W_\Lambda(t)$, where clearly $0 \leq k \leq d-1$. If $k=0$, then $(d,e) \in W_\Lambda(t)$. Suppose $k > 0$. Define the ordered set $W$ of elements of $D$ by $W{=}\{(d{,}e), (d{-}a_e{,}e{+}1), \cdots{,}(d{-}\sum_{i=e}^{c_t-1} a_i{,}c_t)\}$. Each element of $W$ is $k$ rows below an element of $W_\Lambda(t)$, so $W$ must meet or cross $S_\Lambda(t)$, by which time the corresponding entries of $A(\Lambda)$ have become negative. By Lemma 4.13 there must be an element of $W$ for which the corresponding entry of $A(\Lambda)$ is zero, this element must lie to the left of $S_\Lambda(t)$, hence in $D(t)$, i.e., there is $\gamma_s$, $1{\leq}s{\leq}t$, such that $(b_s{,}c_s){\in}W{\cap}D(t)$ and so $(d{,}e){\in}W_\Lambda(s)$. The proof of (ii) is analogous to (i). For (iii), in both cases, $(b_s{,}c_s){\in}D(t)$; suppose $(b_s{,}c_s)$ is $k$ rows below $W_\Lambda(t)$, i.e., $(b_s{-}k{,}c_s){\in}W_\Lambda(t)$, $k{\geq}1$. We then have $b_s{-}k{=}b_t{+}\sum_{i=c_s}^{c_t-1} a_i$. Now, since $A(\Lambda)_{b_t,c_t}{=}0$, we have $x_{st}{=}\sum_{i=c_s}^{c_t-1} a_i{+}c_t{-}c_s{=}b_s{-}b_t{+}c_t{-}c_s{-}k{=}h_{st}{-}1{-}k{<}h_{st}{-}1$. Hence $\gamma_s, \gamma_t$ are $c$-related. ∎

*Lemma 4.15.* If $1 \leq s < t \leq r$ and $\gamma_s, \gamma_{s+1}, \ldots, \gamma_{t-1}$ are all $c$-related to $\gamma_t$, then $D(s) \subset D(t)$.

*Proof.* Let $b{=}b_t{+}\sum_{i=c_s}^{c_t-1} a_i$, by (4.2a), $(b{,}c_s){\in}W_\Lambda^e(t)$. Since $h_{st}{=}c_t{-}c_s{+}b_s{-}b_t{+}1$, $x_{st}{=}c_t{-}c_s{+}\sum_{i=c_s}^{c_t-1} a_i$, so $b = b_t{+}x_{st}{-}c_t{+}c_s < b_t{+}h_{st}{-}1{-}c_t{+}c_s = b_s$. i.e., $(b{,}c_s)$ is above $(b_s{,}c_s)$ and since $A(\Lambda)_{b_s,c_s}{=}0$, so the entry $A(\Lambda)_{b,c_t}{>}0$. Similarly, let $c{=}c_s{-}\sum_{i=b_t}^{b_s-1} a_{\overline{m-i+1}}$, then $(b_s{,}c){\in}S_\Lambda^e(t)$ is to the right of $(b_s{,}c_s)$ and the entry $A(\Lambda)_{b_s,c}{<}0$. Elements $(b,c_s)$, $(b_s,c)$ will be in fact in $W_\Lambda(t), S_\Lambda(t)$, respectively, provided $W_\Lambda^e(t), S_\Lambda^e(t)$ do not cross above and to the right of these elements. Suppose they do cross; then the entries corresponding to elements of $W_\Lambda^e(t)$, $S_\Lambda^e(t)$ must change from, respectively, positive to negative, negative to positive above and to the right of these elements. By Lemma 4.13, this means that $W_\Lambda^e(t), S_\Lambda^e(t)$ contain a common element $(d,e)$ with $A(\Lambda)_{de} = 0$, i.e., there is $\gamma_p$, $s < p < t$ such that $(d,e) = (b_p,c_p)$. By Lemma 4.6, $\gamma_p, \gamma_t$ must be $q$-related, contradiction with the assumption of the lemma. This proves that $(b{,}c_s){\in}W_\Lambda(t)$, $(b_s{,}c){\in}S_\Lambda(t)$. It follows that $(b_s,c_s) \in D(t)$. From this, it follows that $W_\Lambda^e(s), S_\Lambda^e(s)$ must cross before $W_\Lambda^e(t), S_\Lambda^e(t)$ cross, so $SW_\Lambda(s){\subset}D(t)$. Hence also $D(s){\subset}D(t)$. ∎

Lemmas 4.14-5 are illustrated by Example 4.3, where $r = 2, \gamma_1, \gamma_2$ are $c$-related. Clearly, $D(1) \subset D(2)$ and every position of $D(2)$ are in $SW_\Lambda(2)$ or $SW_\Lambda(1)$. Now the definition below tells how to determine the composition factors of $\overline{V}(\Lambda)$ corresponding to the unlinked codes.

*Definition 4.16.* Let $\Sigma^c$ be an unlinked code for $\Lambda$ with non-zero columns $C_{s_1}, ..., C_{s_p}$ where $1 \leq s_1 < ... < s_p \leq r$. Define the subset $D_\Sigma$ of $D$ to be $\phi$ if $p = 0$ or $\cup_{l=1}^{p} SW_\Lambda(s_l)$ otherwise. The weight corresponding to the code $\Sigma^c$ is then defined by $\Sigma = \Lambda - \sum_{\beta \in \widehat{D}_\Sigma} \beta$. ∎

$D_\Sigma, \Sigma$ are uniquely determined by $\Sigma^c$. If $\Sigma^c$=0...0, then $D_\Sigma$=$\phi$, $\Sigma$=$\Lambda$. Note that not all unions of chains correspond to codes. Recall that if, for all $u$, $s \leq u < t$, $\gamma_u$ is $c$-related to $\gamma_t$ then in all codes $\Sigma^c$ with non-zero column $t$ its $s$-th column must contain $t$, i.e., $\gamma_s$ must be wrapped by $\gamma_t$. Hence if $SW_\Lambda(t) \subset D_\Sigma$, then also $SW_\Lambda(u) \subset D_\Sigma$ for all $u$, $s \leq u < t$. Now if $s$ is the smallest number such that for all $u$, $s \leq u < t$, $\gamma_u$ is $c$-related to $\gamma_t$, then Lemmas 4.14-5 show that $D(t) = \cup_{u=s}^{t} SW_\Lambda(u)$. Thus the requirement on codes $\Sigma^c$ with non-zero column $t$ that $\gamma_t$ must wrap all $\gamma_u$, $s \leq u < t$ is equivalent to the requirement that $D(t) \subset D_\Sigma$. A union of south west chains not satisfying this condition cannot correspond to a code. If $\Sigma^c$ is an indecomposable unlinked code, then each non-zero column of the code contains a common number, $t$ say. This means that column $t$ is the rightmost non-zero column of $\Sigma^c$ and $\gamma_t$ wraps all the $\gamma$'s corresponding to the other non-zero columns, so that $D_\Sigma = D(t)$. Conversely, if $D_\Sigma = D(t)$ for some $t$, we may reverse the above argument to show $\Sigma^c$ is indecomposable. We therefore have

**Theorem 4.17.** $\Sigma^c$ is an indecomposable unlinked code $\Leftrightarrow$ there exists a $t, D_\Sigma = D(t)$. ∎

*Example 4.18.* In the case $sl(6/5)$ let $\Lambda = [00020; 0; 0210]$ with $A(\Lambda)$ and $nqc(\Lambda)$ as given in (3.3). The south west chains of $A(\Lambda)$ are as follows:

$$A(\Lambda) = \begin{array}{c} \\ 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ \\ \end{array} \begin{array}{c} \left[\begin{array}{ccccc} 7 & 6 & 3 & 1 & 0 \\ 6 & 5 & 2 & 0 & \bar{1} \\ 5 & 4 & 1 & \bar{1} & \bar{2} \\ 4 & 3 & 0 & \bar{2} & \bar{3} \\ 1 & 0 & \bar{3} & \bar{5} & \bar{6} \\ 0 & \bar{1} & \bar{4} & \bar{6} & \bar{7} \end{array}\right] \\ \begin{array}{cccc} 0 & 2 & 1 & 0 \end{array} \end{array} . \qquad (4.3)$$

All codes were given in (3.4). One decomposable unlinked code is $\Sigma^c = {}^{1\,0\,\substack{3\\4}\,4\,0}$, in which $\gamma_4$ wraps $\gamma_3$. The non-zero columns are the 1st, 3rd, 4th and $D_\Sigma = SW_\Lambda(1) \cup SW_\Lambda(3) \cup SW_\Lambda(4) = D(1) \cup D(4)$, $\Sigma = \Lambda - \beta_{61} - \beta_{43} - (\beta_{33} + \beta_{24} + \beta_{34} + \beta_{44})$. One indecomposable unlinked code is $\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 5 & \\ 5 & & 5 & & \end{smallmatrix}$, in which $\gamma_2$ wraps $\gamma_1$, $\gamma_4$ wraps $\gamma_3$ and $\gamma_5$ wraps all $\gamma_1, ..., \gamma_4$ and $D_\Sigma = \cup_{s=1}^{5} SW_\Lambda(s) = D(5)$ and $\Sigma = \Lambda - \beta_{61} - (\beta_{51} + \beta_{52} + \beta_{62}) - \beta_{43} - (\beta_{33} + \beta_{24} + \beta_{34} + \beta_{44}) - (\beta_{41} + \beta_{42} + \beta_{23} + \beta_{14} + \beta_{15} + \beta_{25} + \beta_{35} + \beta_{45} + \beta_{53} + \beta_{63})$. On the other hand, $SW_\Lambda(2)$ and $SW_\Lambda(1) \cup SW_\Lambda(3) \cup SW_\Lambda(5)$ do not correspond to codes since they violate the necessary conditions that $\gamma_2$ must wrap $\gamma_1$ and $\gamma_5$ must wrap all of $\gamma_1, ..., \gamma_4$. ∎

It was part of the conjecture[2] that for all unlinked codes $\Sigma^c$ of $\Lambda$, $\Sigma$ defined in Definition 4.16 is the highest weight of a composition factor $V(\Sigma)$ of $\overline{V}(\Lambda)$ (note that in *Ref.* 2, $\Sigma$ was

defined by means of boundary strip removals; for unlinked codes it is easy to see that this is equivalent to the definition here). Note that this implies that $\Sigma$ is a dominant weight, which was in fact proved using the corresponding strip removals in the Young diagram. For $\Sigma^c = 0...0, \Sigma = \Lambda$, we see that this code corresponds to the top composition factor $V(\Lambda)$. We shall prove that, for any unlinked code $\Sigma^c$ for $\Lambda$, there exists a primitive vector $v_\Sigma$ and correspondingly a composition factor $V(\Sigma)$ of $\overline{V}(\Lambda)$. To make connection with codes explicit, we use notation $v_{\Sigma^c}$, $U(\Sigma^c)$, $V(\Sigma^c)$ to denote, respectively, $v_\Sigma$, $\mathbf{U}(G)v_\Sigma$, $V(\Sigma)$. Thus, if $\Sigma^c = 0...0$, then $v_{(0...0)} \equiv v_\Lambda, U(0...0) \equiv \overline{V}(\Lambda)$ and $V(0...0) \equiv V(\Lambda)$. Finally, it is worth mentioning that the linked codes for $\Lambda$, if any occur, appear to correspond to south west chains in $A(\Pi^+)$, where $\Pi^+ = \Pi + 2\rho_1$ and $\Pi$ is the lowest $G_{\overline{0}}$-highest weight of the simple module $V(\Sigma)$ corresponding to the code; more details of this are given in *Ref.* 2.

## V. MORE NOTATION AND PRELIMINARY RESULTS

Define a total order on $\Delta$: $\alpha_{ij} < \alpha_{k\ell} \Leftrightarrow j-i < \ell-k$ or $j-i = \ell-k$, $i > k$. It implies that $\beta_{bc} < \beta_{de} \Leftrightarrow c-b < d-e$ or $c-b = d-e$, $b > d$. This total order on $\Delta_1$ corresponds to the sequence of positions signified by $1, 2, \cdots$ in the following $(m+1) \times (n+1)$ matrix, where $\beta_{bc}$ is the root associated with the $(b, c)$-th entry:

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdots \\ 6 & \cdot & \cdot & \cdots \\ 3 & 5 & \cdot & \cdots \\ 1 & 2 & 4 & \cdots \end{pmatrix} \tag{5.1}$$

By Theorem 2.1, choose a basis $B$ of $\mathbf{U}(G_{-1})$: $B = \{b = \prod_{\beta\in S} f(\beta) | S \subset \Delta_1^+\}$, where $f(\beta)$ is a negative root vector corresponding to $\beta$ and the product $\prod_{\beta\in S} f(\beta) = f(\beta_1)\cdots f(\beta_s)$ is written in the *proper order*: $\beta_1 < \cdots < \beta_s$ and $s = |S|$ (the *depth* of $b$). Define a total order on $B$:

$$b > b' = f(\beta'_1)\cdots f(\beta'_{s'}) \;\Leftrightarrow\; s > s' \;\text{ or }\; s = s', \beta_k > \beta'_k,\, \beta_i = \beta'_i \;(1 \le i \le k-1),$$

where $b, b'$ are in proper order. Recall that an element $v \in \overline{V}(\Lambda)$ can be uniquely written as

$$v = b_1 y_1 v_\Lambda + b_2 y_2 v_\Lambda + \cdots = \textstyle\sum_{i=1}^t b_i y_i v_\Lambda,\; b_i \in B, b_1 > b_2 > \cdots,\; 0 \ne y_i \in \mathbf{U}(G_{\overline{0}}^-). \tag{5.2}$$

Clearly $v{=}0 \Leftrightarrow t{=}0$. If $v \ne 0$, we call $b_1 y_1 v_\Lambda$ the *leading term*. A term $b_i y_i v_\Lambda$ is called a *prime term* if $y_i \in \mathbb{C}$. Note that a vector $v$ may have zero or more than one prime terms. One immediately has

*Lemma 5.1.* Let $v{=}gu$, $u \in \overline{V}(\Lambda)$, $g \in \mathbf{U}(G^-)$. (i) If $u$ has no prime term then $v$ has no prime term. (ii) Let $v'{=}gu'$, $u' \in \overline{V}(\Lambda)$. If $u, u'$ have the same prime terms then $v, v'$ have the same prime terms. ∎

*Lemma 5.2.* (i) Let $v_\Sigma \in \overline{V}(\Lambda)$ be a $G_{\overline{0}}$-primitive vector with weight $\Sigma$. Then $\Lambda - \Sigma$ is a sum of distinct positive odd roots, furthermore the leading term $b_1 y_1 v_\Lambda$ of $v_\Sigma$ must be a prime term.

(ii) Suppose $v'_\Sigma = \sum_{i=1}^{t'} (b'_i y'_i) v_\Lambda$ is another $G_{\overline{0}}$-primitive vector with weight $\Sigma$. If all prime terms of $v_\Sigma$ are the same as those of $v'_\Sigma$, then $v_\Sigma = v'_\Sigma$.

*Proof.* (i) Let $v_\Sigma$ be as in (5.2). If $y_1 \notin \mathbb{C}$, then there exists $e_k \in G_{\overline{0}}^+$, $(e_k y_1) v_\Lambda \ne 0$. We have

$$e_k v_\Sigma = b_1(e_k y_1)v_\Lambda + [e_k, b_1]y_1 v_\Lambda + b_2(e_k y_2)v_\Lambda + [e_k, b_2]y_2 v_\Lambda + \cdots,$$

and $[e_k, b] = \sum_{p=1}^{s} f(\beta_1)\cdots[e_k, f(\beta_p)]\cdots f(\beta_s)$ if $b = f(\beta_1)\cdots f(\beta_s)$ and (2.4) gives $[e_k, f(\beta_p)] = 0$ or $\pm f(\beta_q)$ with $\beta_q < \beta_p$. The leading term of $b_i(e_k y_i)v_\Lambda + [e_k, b_i]y_i v_\Lambda$ is $b_i(e_k y_i)v_\Lambda$ if $e_k y_i \neq 0$. It follows that $b_1(e_k y_1)v_\Lambda$ is the leading term of $e_k v_\Sigma$, i.e., $e_k v_\Sigma \neq 0$, contradicting that $v_\Sigma$ is $G_{\overline{0}}$-primitive. So, $y_1 \in \mathbb{C}$ and $\Lambda - \Sigma$ is the weight of $b_1$, a sum of distinct positive odd roots.

(ii) Let $v = v_\Sigma - v'_\Sigma$. If $v \neq 0$ (then it must be $G_{\overline{0}}$-primitive), since its prime terms are all cancelled, $v$ has no prime term, therefore by (i), it is not $G_{\overline{0}}$-primitive, a contradiction. ∎

*Lemma 5.3.* Let $\Sigma$ be a (weakly) primitive weight of $\overline{V}(\Lambda)$. Then $\langle \Lambda + \rho | \Lambda + \rho \rangle = \langle \Sigma + \rho | \Sigma + \rho \rangle$.

*Proof.* Using the Casimir operator $\Omega = 2v^{-1}(\rho) + \sum_i u^i u_i + 2\sum_{\alpha \in \Delta^+} f(\alpha)e(\alpha)$, where $\{u^i\}$ is a basis of $H$, $\{u_i\}$ is its deal basis, $v$ is the isomorphism: $H^* \to H$ derived from $\langle\cdot|\cdot\rangle$, *cf.* that of Lie algebras,[7] we see that $\Omega|_{\overline{V}(\Lambda)} = \langle \Lambda + 2\rho | \Lambda \rangle I|_{\overline{V}(\Lambda)}$. Hence, since $\Sigma$ is the weight of a weakly primitive vector, we have $\langle \Lambda + 2\rho | \Lambda \rangle = \langle \Sigma + 2\rho | \Sigma \rangle$. ∎

Define the following zero weight elements of $\mathbf{U}(G_{\overline{0}})$:

$$\omega_i = \begin{cases} m + 1 + i + h_{\overline{m}i}, & i \in I_1, \\ n + 1 - i + h_{in}, & i \in I_2, \end{cases} \qquad \Omega_i = \begin{cases} 1 - \sum_{\overline{m} \le \overline{k} \le i} f_{\overline{k}i} e_{\overline{k}i}, & i \in I_1, \\ 1 - \sum_{i \le k \le n} f_{ik} e_{ik}, & i \in I_2, \end{cases} \qquad X_i = \omega_i + \Omega_i. \quad (5.3)$$

*Lemma 5.4.* $[\omega_i, \omega_j] = [\omega_i, \Omega_j] = [\Omega_i, \Omega_j] = [X_i, X_j] = 0$ for all $i, j \in I_1 \cup I_2$.

*Proof.* $\omega_i \in \mathbb{C} \oplus H$ and $\Omega_i$ with weight 0 imply the vanishing of the first 2 commutators. The 3rd vanishes trivially if $i \in I_1$, $j \in I_2$ or *vice versa*. Say, $i, j \in I_1$, $i < j$. For each summand of $\Omega_i$ we have $[f_{ki}e_{ki}, \Omega_j] = -[f_{ki}e_{ki}, f_{kj}e_{kj} + f_{i+1,j}e_{i+1,j} + \cdots] = 0$ for $\overline{m} \le k \le i \le j \le \overline{1}$, where, by (2.4), the omitted terms commute with $f_{ki}e_{ki}$ and 2 non-vanishing terms are cancelled. Thus the 3rd vanishes. The vanishing of the first three implies the vanishing of the 4th. ∎

Now, from the definition of $\mathbf{U}(G)$ and the $\mathbb{Z}$-grading of $G$, we can define a projection:

$$\varphi: \mathbf{U}(G) \to \mathbf{U}(G^- \oplus H) \text{ derived from } \mathbf{U}(G) = \mathbf{U}(G^- \oplus H) \oplus \mathbf{U}(G)G^+, \qquad (5.4)$$

where $\mathbf{U}(G)G^+$ is a left ideal of $\mathbf{U}(G)$. For $i \in I_1 \cup I_2, c \in \mathbb{C}$ and $g \in \mathbf{U}(G^- \oplus H)$ define $\chi_{i,c}, \chi_i$: $\mathbf{U}(G^- \oplus H) \to \mathbf{U}(G^- \oplus H)$ by (where $\equiv$ means equal under $\bmod\, \mathbf{U}(G)G^+$)

$$\chi_{i,c} g = \varphi((c + \Omega_i)g) = cg + \varphi(\Omega_i g) \equiv cg + \Omega_i g, \quad \chi_i g = \varphi(X_i g) = \varphi((\omega_i + \Omega_i)g) \equiv X_i g. \quad (5.5)$$

*Lemma 5.5.* The operators $\chi_{i,c}$ and $\chi_{j,c'}$ commute, so do $\chi_i$ and $\chi_j$.

*Proof.* For $g \in \mathbf{U}(G^- \oplus H)$, $\chi_{i,c}\chi_{j,c'}g = \chi_{i,c}(c'g + \varphi(\Omega_j g)) = c(c'g + \varphi(\Omega_j g)) + \varphi(\Omega_i(c'g + \varphi(\Omega_j g))) = cc'g + c\varphi(\Omega_j g) + c'\varphi(\Omega_i g) + \varphi(\Omega_i \varphi(\Omega_j g))$ and $\varphi(\Omega_i \varphi(\Omega_j g)) \equiv \Omega_i \varphi(\Omega_j g) \equiv \Omega_i \Omega_j g$ $\bmod\, \mathbf{U}(G)G^+$ (the 1st formula follows from (5.4), the 2nd from the fact that $\mathbf{U}(G)G^+$ is a left ideal of $\mathbf{U}(G)$ ). This gives the 1st part of the lemma by virtue of the commutativity of $\Omega_i, \Omega_j$. Similarly, $\chi_i \chi_j g = \chi_i \varphi(X_j g) = \varphi(X_i \varphi(X_j g)) \equiv X_i \varphi(X_j g) \equiv X_i X_j g \bmod \mathbf{U}(G)G^+$ and the commutativity of $X_i, X_j$ implies the 2nd part of the lemma. ∎

This Lemma allows us to make the following definitions. For any $J=\{j_1,j_2,\ldots\}\subseteq I_1\cup I_2, C=(c_{j_1},c_{j_2},\ldots)\in \mathbb{C}^{\otimes \# J}, g\in \mathbf{U}(G^-\oplus H)$, let

$$\chi_{J,C}\,g=\prod_{j\in J}\chi_{j,c_j}g=\cdots\chi_{j_2,c_{j_2}}\chi_{j_1,c_{j_1}}g,\qquad \chi_J\,g=\prod_{j\in J}\chi_j g=\cdots\chi_{j_2}\chi_{j_1}g. \tag{5.6}$$

Now we are in a position to establish some important results for the successive application of $\chi_{i,c_i}$ to $f_{\overline{r},s}$ for various $i\in I_1\cup I_2$. From (5.3&5.5) and (2.4), for $\overline{m}\le\overline{r}\le 0\le s\le n$, we have

$$\chi_{\overline{i},c_{\overline{i}}}f_{\overline{r}s}=c_{\overline{i}}f_{\overline{r}s}+f_{\overline{i-1},s}f_{\overline{r}\overline{i}},\ \overline{r}\le\overline{i}<0\ \text{ and }\ \chi_{i,c_i}f_{\overline{r}s}=c_if_{\overline{r}s}-f_{\overline{r},i-1}f_{is},\ 0<i\le s. \tag{5.7}$$

Further application of the commutation relations gives:

$$\begin{aligned}
\chi_{\overline{j},c_{\overline{j}}}\chi_{\overline{i},c_{\overline{i}}}f_{\overline{r}s}&=c_{\overline{j}}c_{\overline{i}}f_{\overline{r}s}+c_{\overline{j}}f_{\overline{i-1},s}f_{\overline{r}\overline{i}}+c_{\overline{i}}f_{\overline{j-1},s}f_{\overline{r}\overline{j}}+f_{\overline{i-1},s}f_{\overline{j-1},\overline{i}}f_{\overline{r}\overline{j}}, && \overline{r}\le\overline{j}<\overline{i}<0,\\
\chi_{\overline{j},c_{\overline{j}}}\chi_{i,c_i}f_{\overline{r}s}&=c_{\overline{j}}c_if_{\overline{r}s}-c_{\overline{j}}f_{\overline{r},i-1}f_{is}+c_if_{\overline{j-1},s}f_{\overline{r}\overline{j}}-f_{\overline{j-1},i-1}f_{\overline{r}\overline{j}}f_{is}, && \overline{r}\le\overline{j}<0<i\le s,\\
\chi_{j,c_j}\chi_{i,c_i}f_{\overline{r}s}&=c_jc_if_{\overline{r}s}-c_jf_{\overline{r},i-1}f_{is}-c_if_{\overline{r},j-1}f_{js}+f_{\overline{r},i-1}f_{i,j-1}f_{js}, && 0<j<i\le s.
\end{aligned}$$

The pattern of terms is becoming clear. The following result may be proved inductively:

*Lemma 5.6.* Let $J=\overline{P}\cup Q$ with $\overline{P}\subseteq\{\overline{r},\ldots,\overline{1}\},\ Q\subseteq\{1,\ldots,s\},\ 1\le r\le m,\ 1\le s\le n.$ Then, with the definition (5.6),

$$\begin{aligned}
\chi_{J,C}f_{\overline{r}s}=\ &\sum_{k=0}^{\#\overline{P}}\sum_{\ell=0}^{\#Q}(-1)^\ell\sum_{\substack{\overline{K}=\{\overline{j_k},\ldots,\overline{j_1}\}\subseteq\overline{P}\\ \overline{j_k}<\cdots<\overline{j_1}}}\ \sum_{\substack{L=\{i_1,\ldots,i_\ell\}\subseteq Q\\ i_1<\cdots<i_\ell}}\ \prod_{\overline{j}\in\overline{P}\setminus\overline{K}}c_{\overline{j}}\prod_{i\in Q\setminus L}c_i\,\cdot\\
&f_{\overline{j_0}i_0}\cdot f_{\overline{j_2-1},\overline{j_1}}f_{\overline{j_3-1},\overline{j_2}}\cdots f_{\overline{r}\overline{j_k}}\cdot f_{i_1,i_2-1}f_{i_2,i_3-1}\cdots f_{i_\ell s}
\end{aligned} \tag{5.8}$$

where $\overline{j_0}=\overline{r}$ or $\overline{j_1-1}$ if $k=0$ or not; $i_0=s$ or $i_1-1$ if $\ell=0$ or not. Similarly, successive application of $\chi_i$ to $f_{\overline{r}s}$ gives results exactly analogous to those of (5.7-8) with $c_i$ replaced by $\omega_i$. ∎

In (5.8) negative root vectors $f_{ij}$ correspond to $\alpha_{ij}\in\Delta^+$ and the products of root vectors have been ordered in such a way that the leftmost factor $f_{\overline{j_0}i_0}$ is a odd vector, while the remaining factors $f_{ij}$ are even. Moreover, in every summand the elements $c_{\overline{j}},c_i$ or $\omega_{\overline{j}},\omega_i$, which lie in $\mathbb{C}\oplus H$, precede an element (a product of $f_{ij}$) of $\mathbf{U}(G^-)$ which in every case have weight $-\alpha_{\overline{r}s}$. From this follows the crucial relationship linking $\chi_J$ and $\chi_{J,C}$. For any weight $\lambda$, define

$$c_i(\lambda)=\begin{cases}\sum_{k=\overline{m}}^{i}\lambda(h_k)+m+i & \text{for } i\in I_1,\\ \sum_{k=i}^{n}\lambda(h_k)+n-i & \text{for } i\in I_2.\end{cases} \tag{5.9}$$

With this notation and Lemma 5.5 in the special case for which $r=m$ and $s=n$, we have

*Corollary 5.7.* Let $v_\lambda$ have weight $\lambda$. Then $\chi_J f_{\overline{m}n}v_\lambda=\chi_{J,C}f_{\overline{m}n}v_\lambda$ with $c_i=c_i(\lambda)$, $i\in J$.

*Proof.* It follows from the above remarks about the order and nature of the factors in (5.8) that for each $i\in J$ the factor $\omega_i$, defined by (5.3), gives rise to a factor $c_i$ in (5.8) with

$$c_i=\begin{cases}m+1+i+\lambda(h_{\overline{m}i})-\alpha_{\overline{m}n}(h_{\overline{m}i}) & \text{for } i\in\overline{P}\subseteq I_2,\\ n+1-i+\lambda(h_{in})-\alpha_{\overline{m}n}(h_{in}) & \text{for } i\in Q\subseteq I_1.\end{cases}$$

But $\alpha_{\overline{m}n}(h_j) = 1$ if $j = \overline{m}$ or $n$; $= 0$ otherwise and $h_{\overline{m}i} = \sum_{k=\overline{m}}^{i} h_k$, $i \in I_1$ and $h_{in} = \sum_{k=i}^{n} h_k$, $i \in I_2$. It follows that $\alpha_{\overline{m}n}(h_{\overline{m}i}) = \alpha_{\overline{m}n}(h_{in}) = 1$ so that $c_i = c_i(\lambda)$ as required. ∎

It is also worth observing that the explicit expansion (5.8) implies:

*Corollary 5.8.* With notation of Lemma 5.6, $\chi_{J,C} f_{\overline{r}s} = \chi_{J,C}^{(r/s)} f_{\overline{r}s}$. ∎

We shall need commutators of $e_i$ with $\chi_{J,C} f_{\overline{r}s}$. More precisely, we shall need the action of such commutators on certain vectors $v_\lambda \in \overline{V}(\Lambda)$. In this case we have:

*Lemma 5.9.* Let $J = \{\overline{p}, \ldots, \overline{1}; 1, \ldots, q\}$, $C = (c_{\overline{p}}, \ldots, c_{\overline{1}}; c_1, \ldots, c_q)$ with $p \le r \le m$, $q \le s \le n$. Let $v_\lambda \in \overline{V}(\Lambda)$ with weight $\lambda$ satisfying

$$\begin{array}{lll} c_{\overline{r}} = \lambda(h_{\overline{r}}) \text{ if } \overline{p} = \overline{r}, & c_1 - c_{\overline{1}} = \lambda(h_0), & c_s = \lambda(h_s) \text{ if } q = s, \\ c_{\overline{i}} - c_{\overline{i+1}} - 1 = \lambda(h_{\overline{i}}) \text{ if } \overline{p} < \overline{i} < 0, & c_i - c_{i+1} - 1 = \lambda(h_i) \text{ if } 0 < i < q, & \end{array} \tag{5.10}$$

and

$$f_{\overline{r},\overline{p+1}} v_\lambda = 0 \quad \text{if} \quad p < r, \qquad f_{q+1,s} v_\lambda = 0 \quad \text{if} \quad q < s. \tag{5.11}$$

Then for all $i \in I$,

$$[e_i, \chi_{J,C} f_{\overline{r}s}] v_\lambda = \begin{cases} \chi_{J,C} f_{\overline{r-1},s} v_\lambda & \text{if} \quad i = \overline{r} < \overline{p}, \\ \chi_{J,C} f_{\overline{r},s-1} v_\lambda & \text{if} \quad i = s > q, \\ 0 & \text{otherwise.} \end{cases} \tag{5.12}$$

*Proof.* The first thing to note is that the only non-vanishing commutators of $e_i$ with negative root vectors appearing in (5.8) are the following:

$$\begin{array}{llll} [e_{\overline{i}}, f_{\overline{a}\overline{i}}] = f_{\overline{a},\overline{i+1}}, & [e_{\overline{i}}, f_{\overline{i}\overline{i}}] = h_{\overline{i}}, & [e_{\overline{i}}, f_{\overline{i}\overline{b}}] = -f_{\overline{i-1},\overline{b}}, & \text{for } \overline{a} < i < b; \quad (5.13a) \\ [e_0, f_{\overline{a}0}] = f_{\overline{a}\overline{1}}, & [e_0, f_{00}] = h_0, & [e_0, f_{0b}] = f_{1b}, & \text{for } \overline{a} < 0 < b; \quad (5.13b) \\ [e_i, f_{ai}] = f_{a,i-1}, & [e_i, f_{ii}] = h_i, & [e_i, f_{ib}] = -f_{i+1,b}, & \text{for } a < i < b, \quad (5.13c) \end{array}$$

Consider first $0 < i < q$. The only non-vanishing contributions to (5.12) arise from terms in (5.8) that do not contain $c_i c_{i-1}$. These can be grouped together in sets of three so that for any fixed $a < i < b$ the sum of each such set contains the common factor $c_i f_{ai} f_{i+1,b} - f_{a,i-1} f_{ii} f_{i+1,b} + c_{i+1} f_{a,i-1} f_{ib}$. Taking the commutator with $e_i$ and using (5.13c) gives $c_i f_{a,i-1} f_{i+1,b} - f_{a,i-1} h_i f_{i+1,b}$ $- c_{i+1} f_{a,i-1} f_{i+1,b} = f_{a,i-1} f_{i+1,b}(c_i - h_i - 1 - c_{i+1})$, which acts to the right on a sequence of $f_{i_x,i_{x+1}-1}$'s and $v_\lambda$. However, $[h_i, f_{i_x,i_{x+1}-1}] = 0$ since $i < b < i_x < i_{x+1}$ and $h_i$ acts finally on $v_\lambda$ to give $\lambda(h_i)$. Thus all terms contain the common factor $c_i - \lambda(h_i) - 1 - c_{i+1}$, which vanishes by virtue of our hypothesis (5.10). The result for $\overline{p} < i < 0$ is obtained in the same way. Similarly, $e_0$ commutes with all terms in (5.8) containing the product $c_{\overline{1}} c_1$. The non-vanishing contributions to (5.12) can again be grouped into sets of 3 terms such that for any fixed $\overline{a}<0<b$ the sum of each such set contains the common factor $c_{\overline{1}} f_{\overline{a}0} f_{1b} + f_{00} f_{\overline{a}\overline{1}} f_{1b} - c_1 f_{\overline{a}\overline{1}} f_{0b}$. Taking the commutator with $e_0$ and using (5.13b) gives $c_{\overline{1}} f_{\overline{a}\overline{1}} f_{1,b} + h_0 f_{\overline{a}\overline{1}} f_{1,b} - c_1 f_{\overline{a}\overline{1}} f_{1b} = f_{\overline{a}\overline{1}} f_{1,b}(c_{\overline{1}} + h_0 - c_1)$. Moreover $h_0$ commutes with everything else to its right to finally act on $v_\lambda$ giving $\lambda(h_0)$. Thus all terms contain the common factor $c_{\overline{1}} + \lambda(h_0) - c_1$, again vanishes by (5.10).

If $i{=}q{=}s$, $e_s$ commutes with all terms in (5.8) other than those which can be paired so as to give the common factor $-c_s f_{as} + f_{a,s-1} f_{ss}$ acting directly on $v_\lambda$. Commutation with $e_s$

gives $-c_s f_{a,s-1} + f_{a,s-1} h_{ss}$ leading to the common factor $-c_s + \lambda(h_s)$, which vanishes. The result for $i{=}\overline{p}{=}\overline{r}$ follows in the same way. If $i{=}q{<}s$, $e_q$ commutes with every term in (5.8) other than those for which $i_\ell{=}q$, but then $[e_q, f_{i_\ell s}]{=}[e_q, f_{qs}]{=}f_{q+1,s}$. Thus every non-vanishing term contains the factor $f_{q+1,s}v_\lambda$ which vanishes by (5.11). The story is the same for $i{=}\overline{p}{>}\overline{r}$.

For $\overline{m}{\le}i{<}\overline{r}$ or $\overline{r}{<}i{<}\overline{p}$ or $q{<}i{<}s$ or $s{<}i{\le}n$ all commutators with $e_i$ vanish since $i$ appears nowhere as a subscript on any $f_{ab}$ appearing in (5.8). This leaves as non-vanishing only 2 special cases $i{=}s{>}q$ and $i{=}\overline{r}{<}\overline{p}$. In the 1st of these the only non-vanishing commutator of $e_s$ with terms in (5.8) is $[e_s, f_{i_\ell s}]{=}f_{i_\ell, s-1}$. This gives the 2nd case of (5.12). Similarly the only non-vanishing commutator of $e_{\overline{r}}$ with the terms in (5.8) is $[e_{\overline{r}}, f_{\overline{r}\overline{i_k}}]{=}f_{\overline{r-1},\overline{i_k}}$, giving the 1st case of (5.12). ∎

Finally we give a rather technical lemma which plays a crucial role in proving results in §6.

*Lemma 5.10.* Given $\overline{r}, s, t$ with $\overline{m} \le \overline{r} \le \overline{1}$, $1 \le s \le t \le n$. Let $J \subseteq \{\overline{r}, \cdots, \overline{1}, 1, \cdots, s\}$, $C = (c_{j_1}, c_{j_2}, ...) \in \mathbb{C}^{\otimes \# J}$, $C(1) = (c_{j_1}{+}1, c_{j_2}{+}1, ...)$. Then

$$X_J \equiv \chi_{J,C} f_{\overline{r}s} \cdot \chi_{J,C(1)} f_{\overline{r}t} + \chi_{J,C} f_{\overline{r}t} \cdot \chi_{J,C(1)} f_{\overline{r}s} = 0, \tag{5.14}$$

where the 2nd term is obtained from the 1st by interchanging $s$ and $t$. Taking $t = s$, we have

$$\chi_{J,C} f_{\overline{r}s} \cdot \chi_{J,C(1)} f_{\overline{r}s} = 0 \tag{5.15}$$

*Proof.* We prove this by induction on $\# J$. If $\# J = 0$, we immediately have $X_J = 0$ since $f_{\overline{r}s}$, $f_{\overline{r}t}$ anti-commute. Suppose now (5.14) holds for $J'$ with $\# J' < \# J$. For $J$, suppose $J \cap I_2 \ne \phi$ (the proof is similar if $J \cap I_1 \ne \phi$). Choose $j \in J$ to be the largest and let $J' = J \backslash \{j\}$. Let $C'$ and $C'(1)$ be respectively the element $C$ and $C(1)$ corresponding to $J'$. Using (5.7) we have

$$\text{the 1st summand of } X_J = \chi_{J',C'}(c_j f_{\overline{r}s} - f_{\overline{r},j-1} f_{js}) \cdot \chi_{J',C'(1)}((c_j{+}1) f_{\overline{r}t} - f_{\overline{r},j-1} f_{jt}). \tag{5.16}$$

Now one may check the validity of the following identities for all $\overline{r} \le i < j \le s$:

$$\begin{aligned}
&\chi_{i,c_i+1}(f_{\overline{r},j-1} f_{js}) = \chi_{i,c_i+1} f_{\overline{r},j-1} \cdot f_{js}, \\
&f_{js} \chi_{i,c_i+1} f_{\overline{r}t} = \chi_{i,c_i+1}(f_{js} f_{\overline{r}t}) = \chi_{i,c_i+1} f_{\overline{r}t} \cdot f_{js}, \\
&f_{js} \chi_{i,c_i+1}(f_{\overline{r},j-1} f_{jt}) = \chi_{i,c_i+1} f_{\overline{r}s} \cdot f_{jt} + \chi_{i,c_i+1} f_{\overline{r},j-1} \cdot f_{js} f_{jt}.
\end{aligned}$$

Using these, (5.16) becomes

$$\begin{aligned}
&c_j(c_j+1)\chi_{J',C'} f_{\overline{r}s} \cdot \chi_{J',C'(1)} f_{\overline{r}t} - c_j \chi_{J',C'} f_{\overline{r}s} \cdot \chi_{J',C'(1)} f_{\overline{r},j-1} \cdot f_{jt} \\
&-(c_j+1)\chi_{J',C'} f_{\overline{r},j-1} \cdot \chi_{J',C'(1)} f_{\overline{r}t} \cdot f_{js} + \chi_{J',C'} f_{\overline{r},j-1} \cdot \chi_{J',C'(1)} f_{\overline{r}s} \cdot f_{jt} \\
&+\chi_{J',C'} f_{\overline{r},j-1} \cdot \chi_{J',C'(1)} f_{\overline{r},j-1} \cdot f_{js} f_{jt}
\end{aligned}$$

Denote these terms by $w_1, \cdots, w_5$, and denote the corresponding terms for the 2nd summand of $X_J$ by $w_6, \cdots, w_{10}$. Then $X_J = \sum_{k=1}^{10} w_k$. By the inductive hypothesis, we have $w_1 + w_6 = w_2 + w_4 + w_8 = w_3 + w_7 + w_9 = w_5 = w_{10} = 0$. Hence $X_J = 0$. ∎

The importance of this lemma lies in the consequences which flow from the special case (5.15).

## VI. PRIMITIVE VECTORS OF THE KAC-MODULE $\overline{V}(\Lambda)$

Let $\Lambda$ be a dominant $r$-fold atypical weight of $G$ with atypical roots $\{\gamma_1,...,\gamma_r\}$. In this section we first prove that to every indecomposable unlinked code $\Sigma^c$ for $\Lambda$ there corresponds a primitive vector $v_{\Sigma}$ of $\overline{V}(\Lambda)$ having weight $\Sigma$. Then we generalize the result to arbitrary unlinked codes.

As a precursor to the proof we first restrict attention to those $\Lambda$ for which $\gamma_r = \alpha_{\overline{m}n} = \beta_{1,n+1}$ and those codes $\Sigma^c$ for which $D_\Sigma = SW_\Lambda(r)$. It follows that the topmost and rightmost position $TR_D$ of $D_\Sigma$ is $(1, n+1)$. Thus from (3.2),

$$A(\Lambda)_{1,n+1} = \sum_{k=\overline{m}}^{0} a_k - \sum_{k=1}^{n} a_k + m - n = 0. \tag{6.1}$$

It is convenient to introduce special labels for some particular roots in $\widehat{D}_\Sigma$. Let $x = \#\{j|(1,j) \in D_\Sigma\}$, $y = \#\{i|i, n+1) \in D_\Sigma\}$ be respectively the number of elements in the topmost row and rightmost column of $D_\Sigma$. Denote the roots associated with the positions in the topmost row of $D_\Sigma$, taken from right to left, by $\eta_1, \cdots, \eta_x$ and the roots in the rightmost column of $D_\Sigma$, from top to bottom, by $\eta'_1, \cdots, \eta'_y$. Thus, $\eta_1 = \alpha_{\overline{m},n}, \cdots, \eta_x = \alpha_{\overline{m},n-x+1}$, $\eta'_1 = \alpha_{\overline{m},n}, \cdots, \eta'_y = \alpha_{\overline{m-y+1},n}$, with $\eta_1 = \eta'_1 = \beta_{1,n+1}$. It should be noted the definitions of $x$ and $y$ are such that:

$$a_{n-x+1} > 0, a_{n-i+1} = 0,\ i = 1, \cdots, x-1 \quad \text{and} \quad a_{\overline{m-y+1}} > 0, a_{\overline{m-i+1}} = 0,\ i = 1, \cdots, y-1. \tag{6.2}$$

All this may be illustrated as follows in our $sl(6/5)$ Example 4.18 with $\Lambda = [00020; 0; 0210]$ encountered in §§3-4. $A(\Lambda)$ was displayed in (3.3) with codes enumerated in (3.4) and chains in (4.3). $\Sigma^c$ is the indecomposable unlinked code $\begin{smallmatrix}1&2&3&4&5\\2&5&4&5&\\5&&5&&\end{smallmatrix}$. Below we have set $D_\Sigma$ alongside $A(\Lambda)$, specifying all positions on the chain $SW_\Lambda(\ell)$ by an entry $\ell$ for $\ell = 1, \cdots, 5$. Here $x = 2, y = 4$ and in the final array we identify the positions in $D_\Sigma$ associated with roots $\eta_i$, $\eta'_i$:

$$A(\Lambda) = \begin{pmatrix} 7&6&3&1&0\\ 6&5&2&0&\overline{1}\\ 5&4&1&\overline{1}&\overline{2}\\ 4&3&0&\overline{2}&\overline{3}\\ 1&0&\overline{3}&\overline{5}&\overline{6}\\ 0&\overline{1}&\overline{4}&\overline{6}&\overline{7} \end{pmatrix}, \qquad D_\Sigma = \begin{pmatrix} \cdot&\cdot&\cdot&5&5\\ \cdot&\cdot&5&4&5\\ \cdot&\cdot&4&4&5\\ 5&5&3&4&5\\ 2&2&5&\cdot&\cdot\\ 1&2&5&\cdot&\cdot \end{pmatrix} = \begin{pmatrix} \cdot&\cdot&\cdot&\eta_2&\eta_1\\ \cdot&\cdot&*&*&\eta'_2\\ \cdot&\cdot&*&*&\eta'_3\\ *&*&*&*&\eta'_4\\ *&*&*&\cdot&\cdot\\ *&*&*&\cdot&\cdot \end{pmatrix}. \tag{6.3}$$

In this example $\eta_1 = \alpha_{\overline{5}4}, \eta_2 = \alpha_{\overline{5}3}$ and $\eta'_1 = \alpha_{\overline{5}4}, \eta'_2 = \alpha_{\overline{4}4}, \eta'_3 = \alpha_{\overline{3}4}, \eta'_4 = \alpha_{\overline{2}4}$. Keeping this example in mind will help understand the proof below.

We shall always suppose $1 \le x \le y$ as the arguments for $1 \le y \le x$ are entirely analogous. Set

$$\begin{aligned} &\Lambda_0 = \Lambda, && \Lambda_k = \Lambda - \textstyle\sum_{j=1}^{k} \eta_j, k = 1, ..., x,\\ &v_0(\Lambda) = v_\Lambda, && v_k(\Lambda) = \chi_J^{(m/n-k+1)} f(\eta_k) v_{k-1}(\Lambda), k = 1, ..., x, \end{aligned} \tag{6.4}$$

with $J = \{\overline{m-y+1}, \ldots, \overline{1}; 1, \ldots, n-x+1\}$. Recall from Convention 2.2 that $\chi_J^{(m/n-k+1)}$ is the operator $\chi_J$ defined for $sl(m+1, n-k+2)$ rather than for $G = sl(m+1/n+1)$.

*Lemma 6.1.* Let $1 \le k \le x \le y$, then with notation (5.6),

$$v_k(\Lambda) = d_k(\Lambda) v_{k-1}(\Lambda),\ d_k(\Lambda) = \chi_{J,C} f_{\overline{m},n-k+1}, \tag{6.5a}$$

$$c_j = c_j(\Lambda) - k + 1, j \in J,\ c_j(\Lambda) = \begin{cases} \sum_{\ell=\overline{m-y+1}}^{j} a_\ell + m + j, & j \in \{\overline{m-y+1}, \ldots, \overline{1}\}, \\ \sum_{\ell=j}^{n-x+1} a_\ell + n - j, & j \in \{1, \ldots, n-x+1\}. \end{cases} \tag{6.5b}$$

*Proof.* Since $v_{k-1}(\Lambda)$ has weight $\Lambda_{k-1}$, it follows from definition (6.4) and Corollary 5.7 that

$$v_k(\Lambda) = \chi_{J,C}^{(m/n-k+1)} f_{\overline{m},n-k+1} v_{k-1}(\Lambda), \qquad c_j = c_j^{(m/n-k+1)}(\Lambda_{k-1}),\ j \in J. \tag{6.6}$$

However, as can be seen from Corollary 5.8 with $r{=}m$, $s{=}n{-}k{+}1$, we have $\chi_{J,C}^{(m/n-k+1)} f_{\overline{m},n-k+1}$ $=\chi_{J,C} f_{\overline{m},n-k+1}$, giving as required (6.5a). Furthermore, note that $\Lambda(h_i){=}a_i, i{\in}I$, the use of (6.2) in (5.9) immediately gives the 2nd equation of (6.5b), so it remains to prove the 1st of (6.5b). However, it follows from definitions (6.4) and (5.9) with $(m/n)$ replaced by $(m/n{-}k{+}1)$ that

$$c_j^{(m/n-k+1)}(\Lambda_{k-1}) = \begin{cases} \sum_{\ell=\overline{m-y+1}}^{j} a_\ell + m - k + 1 + j, & j \in \{\overline{m-y+1}, \ldots, \overline{1}\}, \\ \sum_{\ell=j}^{n-x+1} a_\ell + n - k + 1 - j, & j \in \{1, \ldots, n-x+1\}. \end{cases} \tag{6.7}$$

An inspection of the 2nd of (6.5b) and (6.7) reveals that $c_j^{(m/n-k+1)}(\Lambda_{k-1}) = c_j(\Lambda){-}k{+}1$, $j \in J$. When used in (6.6) this completes the proof of the 1st of (6.5b). ∎

*Corollary 6.2.* Let $1 \le k \le x$, then $v_k(\Lambda) \neq 0$.

*Proof.* Since $a_{\overline{m-y+1}} > 0, a_{n-x+1} > 0$, (6.5b) implies $c_j(\Lambda) > m + j \ge y - 1 \ge x - 1$, $j \in \{\overline{m-y+1}, \ldots, \overline{1}\}$; $c_j(\Lambda) > n - j \ge x - 1$, $j \in \{1, \ldots, n-x+1\}$. It follows that $c_j(\Lambda) - k + 1 > 0$, $j \in J$, $1 \le k \le x$. However, $v_k(\Lambda) = \prod_{1 \le i \le k, j \in J} (c_j(\Lambda) - i + 1) f(\eta_k) \cdots f(\eta_1) v_\Lambda + \cdots$, where the leading term is written in proper order with the ordering (5.1). Thus $v_k(\Lambda) \neq 0$. ∎

Returning to $sl(6/5)$ Example 4.18 with $\Lambda = [0002;0;0210]$, $x = 2, y = 4$, $J = \{\overline{1};1,2,3\}$. $\Lambda_1 = [\overline{1}002;0;021\overline{1}]$, $\Lambda_2 = \Lambda_x = [\overline{2}002;0;0200]$. The coefficients $c_j(\Lambda)$, $j \in J$ are $(5;6,5,2)$ and $c_j$ associated with $v_1(\Lambda)$, $v_2(\Lambda) = v_x(\Lambda)$ are $(5;6,5,2)$, $(4;5,4,1)$, respectively. These are all non-zero, in accordance with Corollary 6.2. A simpler example is Example 4.5, $\Lambda = [1211;\overline{1};10002]$, $x = y = 1$, $J = \{\overline{4},\overline{3},\overline{2},\overline{1};1,2,3,4,5\}$, $\Lambda_1 = \Lambda_x = [0211;\overline{1};10001]$ and $c_j(\Lambda)$ associated with $v_1(\Lambda) = v_x(\Lambda)$ are $(1,4,6,8;7,5,4,3,2)$, again all non-zero. To discuss the $G$-primitivity of $v_x(\Lambda)$, it should be noted that in the 1st of these 2 examples the weight $\Lambda_x$ is not $G$-dominant, although the restriction of this weight to $G^{(m-1/n)} = sl(5/5)$ is $G^{(m-1/n)}$-dominant. In contrast to this, in the 2nd example $\Lambda_x$ is $G$-dominant. Guided by this distinction between our 2 examples, it is convenient to deal first with a special case:

*Lemma 6.3.* $v_1(\Lambda)$ is a $G$-primitive vector if $x = y = 1$.

*Proof.* In this case, $J = \{\overline{m}, \ldots, \overline{1}; 1, \ldots, n\}$ and from (6.5) we have

$$d_1(\Lambda) {=} \chi_{J,C} f_{\overline{m}n}, \quad c_j {=} c_j(\Lambda) {=} \begin{cases} \sum_{k=\overline{m}}^{j} a_k + m + j, & j \in \{\overline{m}, \ldots, \overline{1}\}, \\ \sum_{k=j}^{n} a_k + n - j, & j \in \{1, \ldots, n\}. \end{cases}$$

It then follows that:

$$\begin{array}{ll} c_{\overline{m}} = a_{\overline{m}}, & c_n = a_n, \\ c_i - c_{i-1} - 1 = a_i \ \text{ if } \ \overline{m} < i < 0, & c_i - c_{i+1} - 1 = a_i \ \text{ if } \ 0 < i < n, \\ \multicolumn{2}{l}{c_1 - c_{\overline{1}} = \sum_{k=1}^{n} a_k + n - 1 - \sum_{k=\overline{m}}^{\overline{1}} a_k - m + 1 = a_0,} \end{array}$$

where, the recovery of $a_0$ in the last case is a consequence of (6.1). It is only here that the

atypicality condition makes itself felt. Since $\Lambda(h_i) = a_i$, $i \in I$, it follows that $v_\Lambda \in \overline{V}(\Lambda)$ satisfies all hypotheses (5.10) of Lemma 5.9 for $p = r = m$, $q = s = n$. The hypotheses (5.11) are redundant, as are the first 2 cases of (5.12). Therefore, we conclude from (5.12) that $[e_i, d_1(\Lambda)]v_\Lambda = 0$, $i \in I$. Since $v_\Lambda$ is itself $G$-primitive, so $d_1(\Lambda)v_\Lambda$ is also $G$-primitive. ∎

Prior to tackling other cases it is convenient to introduce one further preliminary result:

*Lemma 6.4.* $f_{\overline{m},\overline{m-y+2}}v_\Lambda = 0$ if $1 < y$ and $f_{n-x+2,n-k+1}v_\Lambda = 0$ if $1 \le k < x$.

*Proof.* Let $I_{\overline{y},x}=I_{\overline{y}}\cup I_x$, $I_{\overline{y}}=\{\overline{m},...,\overline{m-y+2}\}$, $I_x=\{n-x+2,...,n\}$. Since $e_iv_\Lambda=0$ and (6.2) gives $h_iv_\Lambda =0, i\in I_{\overline{y},x}$, consideration of algebra Span$\{e_i,f_i,h_i\}$ implies that $f_iv_\Lambda=0$. By (2.4), $f_{ij}, i,j\in I_{\overline{y}}$, $f_{k\ell}, k,\ell\in I_x$ can respectively be expressed in terms of $f_i$, $i\in I_{\overline{y}}$, $f_j$, $j\in I_x$. The result then follows. ∎

*Lemma 6.5.* If $1 < x \le y$, then $v_x(\Lambda)$ is a $G^{(m-1/n)} = sl(m/n+1)$ primitive vector.

*Proof.* In this case, by (6.5b), we have

$$\begin{array}{ll} c_{\overline{m-y+1}} = a_{\overline{m-y+1}} + y - k, & c_{n-x+1} = a_{n-x+1} + x - k, \\ c_i - c_{i-1} - 1 = a_i,\ \overline{m-y+1} < i < 0, & c_i - c_{i+1} - 1 = a_i,\ 0 < i < n-x+1, \\ c_1 - c_{\overline{1}} = \sum_{k=1}^{n} a_k + n - 1 - \sum_{k=\overline{m}}^{\overline{1}} a_k - m + 1 = a_0, & \end{array} \tag{6.8}$$

We are going to exploit Lemma 5.9 for $p=m-y+1, q=n-x+1, r=m, s=n-k+1$ with $1<x\le y$, $1\le k\le x$, $v_\lambda=v_{k-1}(\Lambda)$. It is necessary to check that all hypotheses (5.10-11) are satisfied. Noted that $\lambda(h_i)=\Lambda_{k-1}(h_i)=a_i, \overline{m-1}\le i\le n-k+1$ implies $\lambda(h_i)=\Lambda_{k-1}(h_i)=a_i$, $\overline{m-y+1}\le i\le n-x+1$. It follows from (6.8) that the hypotheses (5.10) are all satisfied unless either $i=\overline{p}=\overline{m-y+1}=\overline{r}=\overline{m}$ with $y\ne k$, or $i=q=n-x+1=s=n-k+1$ with $x\ne k$. Neither case can occur. Hence (5.10) is satisfied. It remains to consider (5.11). From (2.4) $[f_{\overline{m},\overline{m-y+2}},f_{ab}]=0$ unless $a=\overline{m-y+1}$ or $b=\overline{m+1}$; $[f_{n-x+2,n-k+1},f_{ab}]=0$ unless $a=n-k+1$ or $b=n-x+1$. However, the expansion of $d_i(\Lambda)$ by means of (5.8) involves only those $f_{ab}$ for which $a\in\{\overline{m}\}\cup\{\overline{m-y},...,n-x+1\}$ and $b\in\{\overline{m-y+1},...,n-x\}\cup\{n-i+1\}$ . This implies for $i=1,...,k-1$ we have $[f_{\overline{m},\overline{m-y+2}},d_i(\Lambda)]=0, 1<y$, $[f_{n-x+2,n-k+1},d_i(\Lambda)]=0, 1\le k<x$. Since $v_{k-1}(\Lambda)=d_{k-1}(\Lambda)\cdots d_1(\Lambda)v_\Lambda$ it follows from Lemma 6.4 that $f_{\overline{m},\overline{m-y+2}}v_{k-1}(\Lambda)=0$, $f_{n-x+2,n-k+1}v_{k-1}(\Lambda)=0$, confirming that (5.11) is satisfied. Lemma 5.9 then implies that $[e_i, d_k(\Lambda)]v_{k-1}(\Lambda) = 0$ for $i \in I, 1 \le k \le x$ unless either $i = \overline{m}$ or $i = n-k+1, 1 \le k < x$. The 1st case does not concern us within $G^{(m-1/n)}$. The other cases imply that since $v_x(\Lambda) = g_xv_\Lambda$, $g_x = d_x(\Lambda)\cdots d_1(\Lambda)$ we have $e_iv_x(\Lambda) = g_xe_iv_\Lambda = 0$ unless $i = n-k+1, 1 \le k < x$. If $i = n-k+1, 1 \le k < x$ we have

$$\begin{aligned} e_{n-k+1}v_x(\Lambda) &= d_x(\Lambda)\cdots d_{k+1}(\Lambda)e_{n-k+1}d_k(\Lambda)v_{k-1}(\Lambda) \\ &= d_x(\Lambda)\cdots d_{k+1}(\Lambda)[e_{n-k+1}, d_k(\Lambda)]v_{k-1}(\Lambda) + d_x(\Lambda)\cdots d_1(\Lambda)e_{n-k+1}v_\Lambda \end{aligned}$$

However, $e_{n-k+1}v_\Lambda = 0$. Furthermore, the 2nd case of (5.12) and definitions (5.6& 6.6) give

$$d_{k+1}(\Lambda)[e_{n-k+1}, d_k(\Lambda)]v_{k-1}(\Lambda) = \prod_{j\in J} \chi_{j,c_j(\Lambda)-k} f_{\overline{m},n-k} \prod_{j\in J} \chi_{j,c_j(\Lambda)-k+1} f_{\overline{m},n-k} v_{k-1}(\Lambda) = 0,$$

where the final equality is a consequence of (5.15) in Lemma 5.10. We conclude that $e_{n-k+1}v_x(\Lambda) = 0$ for $1 \le k < x$, thereby completing the proof that $v_x(\Lambda)$ is $G^{(m-1/n)}$-primitive. ∎

**Theorem 6.6.** To any indecomposable unlinked code $\Sigma^c$ for $\Lambda$, there corresponds a primitive vector $v_\Sigma = g_\Sigma v_\Lambda$ of $\overline{V}(\Lambda)$ with weight $\Sigma$ for some $g_\Sigma \in \mathbf{U}(G^-)$.

*Proof.* Suppose that the topmost and rightmost position $TR_D$ of $D_\Sigma$ is $(m+1-m_\Sigma, n_\Sigma+1)$. By Theorem 4.17 there exists $t$ with $1 \le t \le r$ such that $\gamma_t = \alpha_{\overline{m_\Sigma}, n_\Sigma}$, $1 \le m_\Sigma \le m$, $1 \le n_\Sigma \le n$. First we suppose that $m_\Sigma = m, n_\Sigma = n$ so that $t = r, \gamma_t = \alpha_{\overline{m},n}, TR_D = (1, n+1)$. We shall see later, by restriction from $G^{(m/n)}$ to $G^{(m_\Sigma/n_\Sigma)}$, we can prove the theorem in general. Under this assumption, we are going to prove it by induction on the depth $d = \#\widehat{D}_\Sigma$. For $d = 1$ for which $D_\Sigma$ necessarily consists of the single position $(1, n+1)$ and $x = y = 1$. Thus, $\Sigma = \Lambda - \alpha_{\overline{m},n}$ is precisely the weight of the vector $v_1(\Lambda) = d_1(\Lambda)v_\Lambda$ which was shown to be $G$-primitive in Lemma 6.3 and our Theorem 6.6 is satisfied by $v_\Sigma = v_1(\Lambda)$.

Let $d > 1$. With notation (6.4), by Lemma 6.5, $v_x(\Lambda)$ is $G^{(m-1/n)}$-primitive with weight $\Lambda_x$. The atypicality matrix $A(\Lambda_x)$ is obtained from $A(\Lambda)$ by subtracting $x$ from all entries in the topmost row and adding 1 from each of the last $x$ columns. By removing the topmost row of $A(\Lambda_x)$ we obtain the atypicality matrix $A^{(m-1/n)}(\Lambda_x)$ of $\Lambda_x$ restricted to $G^{(m-1/n)}$. It is easy to check that the $(2, n+1)$-th entry in $A(\Lambda_x)$ is always zero, so that $\beta_{2,n+1} = \alpha_{\overline{m-1},n}$ is an atypical root for $\Lambda_x$ restricted to $G^{(m-1/n)}$. Moreover, $\Lambda_x$ is $r_x$-fold atypical with respect to $G^{(m-1/n)}$ where $r_x = r$ if $x < y$ or $r-1$ if $x = y$. Using results in §4, one can then see that there is an indecomposable unlinked code $\Sigma^c_x$ for the restriction of $\Lambda_x$ to $G^{(m-1/n)}$ for which

$$\widehat{D}_{\Sigma_x} = \widehat{D}_\Sigma^{(m-1/n)} = \widehat{D}_\Sigma \backslash \{\eta_1, ..., \eta_x\} = \widehat{D}(r_x)^{(m-1/n)}, \tag{6.9}$$

with the topmost and rightmost position of $D_\Sigma^{(m-1/n)}$ being $(1, n+1)$ in $A(\Lambda_x)^{(m-1/n)}$ (but which is position $(2,n+1)$ in $A(\Lambda_x)$). This may again be illustrated by $sl(6/5)$ example with $\Lambda=[00020; 0; 0210]$, $x=2$, $\Lambda_x=[\overline{1}0020; 0; 0200]$. On restriction this yields $sl(5/5)$-dominant weight $\Lambda_x=[0020; 0; 0200]$ for which there exists an indecomposable unlinked code $\Sigma^c_x$ again given by $\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 5 & \\ 5 & & 5 & & \end{smallmatrix}$. The corresponding atypicality matrix $A(\Lambda_x)$ and $D_{\Sigma_x}$ take the form (*cf.* (6.3)):

$$A(\Lambda_x) = \begin{pmatrix} 5 & 4 & 1 & 0 & \overline{1} \\ 6 & 5 & 2 & 1 & 0 \\ 5 & 4 & 1 & 0 & \overline{1} \\ 4 & 3 & 0 & \overline{1} & \overline{2} \\ 1 & 0 & \overline{3} & \overline{4} & \overline{5} \\ 0 & \overline{1} & \overline{4} & \overline{5} & \overline{6} \end{pmatrix}, \qquad D_{\Sigma_x} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 5 & 5 & 5 \\ \cdot & \cdot & 4 & 4 & 5 \\ 5 & 5 & 3 & 4 & 5 \\ 2 & 2 & 5 & \cdot & \cdot \\ 1 & 2 & 5 & \cdot & \cdot \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & * & * & \eta_2' \\ \cdot & \cdot & * & * & \eta_3' \\ * & * & * & * & \eta_4' \\ * & * & * & \cdot & \cdot \\ * & * & * & \cdot & \cdot \end{pmatrix}.$$

The positions of the entries $*$ serve to specify the roots $\beta \in \widehat{D}_\Sigma^{(4/4)}$. The result conforms precisely with (6.9) as can be seen by comparison with the diagrams specifying the roots $\beta \in D_\Sigma$ in (6.3). Now let $U(\Lambda_x)$ be the cyclic $G^{(m-1/n)}$-submodule $\mathbf{U}(G^{(m-1/n)})v_x(\Lambda)$, which turns out to be isomorphic to the Kac-module $\overline{V}(\Lambda_x)^{(m-1/n)}$ of $G^{(m-1/n)}$. Now the depth of $\Sigma$ relative to $\Lambda_x$ is given by $d_x = \#\widehat{D}_\Sigma^{(m-1/n)} = \#\widehat{D}_\Sigma - x = d - x < d$, by induction hypothesis we see that there must exist some $g_\Sigma^{(m-1/n)} \in \mathbf{U}((G^-)^{(m-1/n)})$ such that $0 \ne v_\Sigma = g_\Sigma^{(m-1/n)} v_x(\Lambda) = g_\Sigma v_\Lambda$ is $G^{(m-1/n)}$-primitive, with $g_\Sigma = g_\Sigma^{(m-1/n)} g_x \in \mathbf{U}(G^-)$, $g_x = d_x(\Lambda) \cdots d_1(\Lambda)$.

*Lemma 6.7.* $v_\Sigma$ defined above is $G$-primitive if $x < y$.

*Proof.* It remains to prove $e_{\overline{m}} v_\Sigma = 0$. First, in constructing $A(\Sigma)$ from $A(\Lambda)$ with $\Sigma = \Lambda - \sum_{\beta \in \widehat{D}_\Sigma} \beta$, $A(\Lambda)_{1,n+1} = 0$, we have $A(\Sigma)_{1,n+1} = y - x > 0$. Since $\Sigma$ is dominant we have

$A(\Sigma)_{1,i+1} > A(\Sigma)_{1,i+2}$ for $i = 0, \dots, n-1$. Hence, from (3.1a) we have

$$A(\Sigma)_{1,i+1} = \langle \Sigma + \rho | \alpha_{\overline{m}i} \rangle > 0 \quad \text{for} \quad i = 0, \dots, n. \tag{6.10}$$

Second, we can write $\Sigma = \Lambda - \sum_{\beta \in \widehat{D}_\Sigma} \beta = \Lambda - \sum_{i=1}^{d} \beta_j$ in the way that $\beta_i$ is an atypical root of $\Sigma_{i-1} = \Lambda - \sum_{j=1}^{i-1} \beta_j$ (which is not necessarily $G$-dominant) in the sense that $\langle \Sigma_{i-1} + \rho | \beta_i \rangle = 0$. Induction on $i$ gives $\langle \Sigma_i + \rho | \Sigma_i + \rho \rangle = \langle \Lambda + \rho | \Lambda + \rho \rangle$. When $i = d$ we obtain $\langle \Sigma + \rho | \Sigma + \rho \rangle = \langle \Lambda + \rho | \Lambda + \rho \rangle$. Now suppose that $e_{\overline{m}} v_\Sigma \neq 0$. Let $g$ be an element in $\mathbf{U}(G^+)$ with the largest possible weight $\mu$ such that $u = g v_\Sigma \neq 0$. By this definition, $u$ is $G$-primitive with weight $\Sigma' = \Sigma + \mu$. Using that $v_\Sigma$ is $G^{(m-1/n)}$-primitive, $g$ can be chosen to be a sum of the form

$$g = e_{\overline{m},i_1} \cdots e_{\overline{m},i_k}, \quad \text{with} \quad \mu = \textstyle\sum_{j=i_1}^{i_k} \alpha_{\overline{m}j}, \tag{6.11}$$

where $i_1 \leq \cdots \leq i_k$ and $i_j < \cdots < i_k$ if $i_j \geq 0$. Since $u$ has weight $\Sigma + \mu$, Lemma 5.3 gives $\langle \Sigma + \mu + \rho | \Sigma + \mu + \rho \rangle = \langle \Lambda + \rho | \Lambda + \rho \rangle$, combined with $\langle \Sigma + \rho | \Sigma + \rho \rangle = \langle \Lambda + \rho | \Lambda + \rho \rangle$, we obtain

$$2\langle \Sigma + \rho | \mu \rangle + \langle \mu | \mu \rangle = 0. \tag{6.12}$$

Since $\mu$ has the form (6.11), denote it by $\mu_k$. Induction on $k$ gives $\langle \mu_k | \mu_k \rangle = \langle \mu_{k-1} | \mu_{k-1} \rangle + \langle 2\mu_{k-1} + \alpha_{\overline{m},i_k} | \alpha_{\overline{m},i_k} \rangle \geq 0$, where for the last inner product, we have made use of $\langle \alpha_{\overline{m}i} | \alpha_{\overline{m}j} \rangle \geq 0$ for $i, j \in I$ (this can easily be proved by (2.2)). Also we may prove as follows that $\langle \Sigma + \rho | \alpha_{\overline{m}i} \rangle > 0$ for all $i \in I$: if $i < 0$, then $\alpha_{\overline{m}i}$ is a positive even root, so $\langle \Sigma | \alpha_{\overline{m}i} \rangle \geq 0$, $\langle \rho | \alpha_{\overline{m}i} \rangle > 0$; if $i \geq 0$, then $\alpha_{\overline{m}i}$ is a positive odd root and $\langle \Sigma + \rho | \alpha_{\overline{m}i} \rangle > 0$ by (6.10). This proves that the *l.h.s.* of (6.12) is $> 0$; this contradiction proves that $e_{\overline{m}} v_\Sigma$ must be zero. Hence the lemma follows. ∎

*Lemma 6.8.* Let $x{=}y$. If $v_{\Sigma}$ is not $G$-primitive then $e_{\overline{m}n} v_{\Sigma}$ is primitive and $e_{\overline{m}i} v_{\Sigma} \neq 0$, $i \leq n$.

*Proof.* The proof is very similar to that of Lemma 6.7. If $v_{\Sigma}$ is not primitive, we want to prove that $g$ in (6.11) must be $e_{\overline{m}n}$. The only difference is that now $A(\Sigma)_{1,n+1} {=} y {-} x {=} 0$. Thus (6.10) must be replaced by the statement $\langle \Sigma{+}\rho | \alpha_{\overline{m}i} \rangle {>} 0, i{=}0,...,n\text{-}1, \langle \Sigma{+}\rho | \alpha_{\overline{m}n} \rangle {=} 0$. Hence, by the same argument as before, for our hypothesized $G$-primitive vector $u{=}g v_{\Sigma}, g$ must have weight $\mu{=}\alpha_{\overline{m}n}$, since this is the only possible solution of (6.12). It follows that $g{=}e_{\overline{m}n}$. By our choice of $g$, $e_{\overline{m}n} v_{\Sigma} \neq 0$ and so $e_{\overline{m}i} v_{\Sigma} {\neq} 0$ since $e_{\overline{m}n} v_{\Sigma} {=} {-} e_{in} e_{\overline{m}i} v_{\Sigma}$ for $i < n$. ∎

By Lemma 6.7, if $x < y$, the proof of the theorem is then completed. So, let $x{=}y$. If $v_{\Sigma}$ is $G$-primitive, the proof is also completed. Suppose now $v_{\Sigma}$ is not $G$-primitive. First note that

$$\begin{aligned} \widehat{D}_{\Sigma} &= \widehat{D}_{\Sigma}^{(m-1/n)} \cup \{\eta_1, ..., \eta_x\} = \widehat{D}_{\Sigma}^{(m/n-1)} \cup \{\eta'_1, ..., \eta'_x\} \\ &= \widehat{D}_{\Sigma}^{(m-1/n-1)} \cup \{\alpha_{\overline{m}n}\} \cup \{\eta_2, ..., \eta_x\} \cup \{\eta'_2, ..., \eta'_x\}. \end{aligned} \tag{6.13}$$

Then we see that $v_{\Sigma}$ defined above can be written in the form

$$v_{\Sigma} = g_{\Sigma} v_{\Lambda} = g_{\Sigma}^{(m-1/n)} v_x(\Lambda) = g_{\Sigma}^{(m-1/n)} g_x v_{\Lambda} = g_{\Sigma}^{(m-1/n-1)} g'_{x-1} g_x v_{\Lambda}, \tag{6.14}$$

where, quite generally the weight $\text{wt}(g_{\Sigma}^{(r/s)}) {=} {-} \sum_{\alpha \in \widehat{D}_{\Sigma}^{(r/s)}} \alpha$ and $\text{wt}(g_x) {=} {-} \sum_{i=1}^{x} \eta_i$, $\text{wt}(g'_{x-1}) {=} {-} \sum_{i=2}^{x} \eta'_i$. Furthermore, by induction on $\# D_{\Sigma}$, (6.13-14) tells that we can decompose

$$D_{\Sigma} {=} \cup_{i=0}^{i_{\Sigma}} D^{(i)}, \qquad D^{(1)} {=} \{\eta'_2, ..., \eta'_x\}, \qquad D^{(0)} {=} \{\eta_1, ..., \eta_x\}, \tag{6.15a}$$

$$g_{\Sigma} {=} g^{(i_{\Sigma})} ... g^{(1)} g^{(0)}, \quad g_{\Sigma}^{(m-1/n-1)} {=} g^{(i_{\Sigma})} ... g^{(2)}, \quad g^{(1)} {=} g'_{x-1}, \;\; g^{(0)} {=} g_x, \tag{6.15b}$$

for some $i_{\Sigma}$ such that $\mathrm{wt}(g^{(i)}){=}{-}\sum_{\alpha\in\widehat{D}^{(i)}}\alpha$, where $D^{(i)}$ consists all positions of either the topmost row or the rightmost column in $D_\Sigma\backslash\cup_{j=0}^{i-1}D^{(j)}$. Now suppose the position $P_x{=}(x,n{+}1{-}x)$, which is clearly in $D_\Sigma$, belongs to $D_\Sigma^{(j_\Sigma)}$ for some $j_{\Sigma}$: $2{\le}j_{\Sigma}{\le}i_{\Sigma}$, then we can write

$$g_{\Sigma}=g_{\Sigma}^{(2)}g_{\Sigma}^{(1)},\qquad g_{\Sigma}^{(2)}=g^{(i_\Sigma)}\cdots g^{(j_\Sigma+1)},\quad g_{\Sigma}^{(1)}=g^{(j_\Sigma)}\cdots g^{(0)},\tag{6.15c}$$

$$D_\Sigma=D_\Sigma^{(2)}\cup D_\Sigma^{(1)},\quad D_\Sigma^{(2)}=\cup_{i=j_\Sigma+1}^{i_\Sigma}D^{(i)},\quad D_\Sigma^{(1)}=\cup_{i=0}^{j_\Sigma}D^{(i)},\tag{6.15d}$$

with $\mathrm{wt}(g_{\Sigma}^{(i)})=\sum_{\alpha\in D_\Sigma^{(i)}}\alpha$, $i=1,2$. As an example, consider $sl(5/6)$ with $\Lambda=[0011;1;00200]$, $\Sigma^c{=}\begin{smallmatrix}1&2&3&4\\3&3&4\\4&4\end{smallmatrix}$. Below we have set $D_\Sigma$ alongside $A(\Lambda)$, specifying all positions in $D^{(i)}$ by an entry $i$ for $i{=}0,\dots,8$. Here $x{=}y{=}3$, $i_{\Sigma}{=}8$, $j_{\Sigma}{=}4$. In the final array we identify the positions in $D_\Sigma^{(1)}$ by 1.

$$A(\Lambda)=\begin{pmatrix}7&6&5&2&1&0\\6&5&4&1&0&\overline{1}\\5&4&3&0&\overline{1}&\overline{2}\\3&2&1&\overline{2}&\overline{3}&\overline{4}\\1&0&\overline{1}&\overline{4}&\overline{5}&\overline{6}\end{pmatrix},\ \ D_\Sigma=\begin{pmatrix}.&.&.&0&0&0\\.&.&.&2&2&1\\4&4&4&4&3&1\\7&7&6&5&3&.\\8&8&6&5&.&.\end{pmatrix}=\begin{pmatrix}.&.&.&1&1&1\\.&.&.&1&1&1\\1&1&1&1&1&1\\*&*&*&*&1&.\\*&*&*&*&.&.\end{pmatrix}.\tag{6.16}$$

From this example, it it easy to obtain the following result.

*Lemma 6.9.* (i) All positions above and to the right of $P_x$ are in $D_\Sigma^{(1)}$, i.e., $(i,j){\in}D_\Sigma^{(1)}$, $1{\le}i{\le}x$, $n{+}1{-}x{\le}j{\le}n{+}1$; none position below and to the left of $P_x$ is in $D_\Sigma^{(1)}$, i.e., $(i,j){\notin}D_\Sigma^{(1)}$, $i{>}x,j{<}n{+}1{-}x$;

(ii)(a) If $(i,j){\in}D_\Sigma^{(1)}$, $i{\le}x$ then $(i,j'){\in}D_\Sigma^{(1)}$ for all $j'$: $j{\le}j'{\le}n{+}1$. (b) If $(i,j){\in}D_\Sigma^{(1)}$, $n{+}1{-}x{\le}j$ then $(i',j){\in}D_\Sigma^{(1)}$ for all $i'$: $1{\le}i'{\le}i$. ∎

Now remember that $x{=}y$, we can construct another vector $\widetilde{v}_x(\Lambda)$ if we start from the last column of $D_\Sigma$ instead of the 1st row, such that $\widetilde{v}_x(\Lambda){=}\widetilde{g}_xv_{\Lambda}$, which has similar properties to those of $v_x(\Lambda)$ but where $\mathrm{wt}(\widetilde{g}_x){=}{-}\sum_{i=1}^x\eta'_i$. Then from $\widetilde{v}_x(\Lambda)$, we can construct a vector

$$\widetilde{v}_{\Sigma}=\widetilde{g}_{\Sigma}v_{\Lambda}=\widetilde{g}_{\Sigma}^{(m/n-1)}\widetilde{v}_x(\Lambda)=\widetilde{g}_{\Sigma}^{(m/n-1)}\widetilde{g}_xv_{\Lambda}=\widetilde{g}_{\Sigma}^{(m-1/n-1)}\widetilde{g}'_{x-1}\widetilde{g}_xv_{\Lambda},\tag{6.17}$$

which is $G^{(m/n-1)}$-primitive and $\mathrm{wt}(\widetilde{g}'_{x-1}){=}{-}\sum_{i=2}^x\eta_i$. Note that both $g_{\Sigma}^{(m-1/n-1)}$, $\widetilde{g}_{\Sigma}^{(m-1/n-1)}$ are the element $g_{\Sigma_1^{(m-1/n-1)}}$ defined for the weight $\Sigma_1{=}\Lambda{-}\sum_{i=1}^x\eta_i{-}\sum_{i=2}^x\eta'_i$ restricted to $G^{(m-1/n-1)}$, so, by induction on $m,n$, we can suppose $g_{\Sigma}^{(m-1/n-1)}{=}\widetilde{g}_{\Sigma}^{(m-1/n-1)}$ and so by (6.13-15), we have

$$v_{\Sigma}=g_{\Sigma}^{(2)}g_{\Sigma}^{(1)}v_{\Lambda},\quad\text{and}\quad\widetilde{v}_{\Sigma}=g_{\Sigma}^{(2)}\widetilde{g}_{\Sigma}^{(1)}v_{\Lambda},\ \widetilde{g}_{\Sigma}^{(1)}=g^{(j_\Sigma)}\cdots g^{(2)}\widetilde{g}'_{x-1}\widetilde{g}_x\tag{6.18}$$

Now if $\widetilde{v}_{\Sigma}$ is $G$-primitive, then the proof is again completed, or by analogy with Lemma 6.8, $e_{\overline{m}n}\widetilde{v}_{\Sigma}$ is $G$-primitive. Thus, we can suppose

$$e_{\overline{m}}v_{\Sigma}\ne0\ne e_n\widetilde{v}_{\Sigma},\quad\text{but both } e_{\overline{m}n}v_{\Sigma}\text{ and }e_{\overline{m}n}\widetilde{v}_{\Sigma}\text{ are }G\text{-primitive.}\tag{6.19}$$

*Lemma 6.10.* Let $v_{\Sigma}$ and $\widetilde{v}_{\Sigma}$ be as in (6.19). Then $e_{\overline{m}n}v_{\Sigma}=e_{\overline{m}n}\widetilde{v}_{\Sigma}$ (up to a non-zero scalar).

*Proof.* By (6.18), we have

$$e_{\overline{m}n}v_{\Sigma}=g_{\Sigma}^{(2)}v_2,\ v_2=u_2v_{\Lambda}\quad\text{and}\quad e_{\overline{m}n}\widetilde{v}_{\Sigma}=g_{\Sigma}^{(2)}\widetilde{v}_2,\ \widetilde{v}_2=\widetilde{u}_2v_{\Lambda},\tag{6.20}$$

such that $u_2{=}[e_{\overline{m}n},g_{\Sigma}^{(1)}]$, $\widetilde{u}_2{=}[e_{\overline{m}n},\widetilde{g}_{\Sigma}^{(1)}]$ have the weight $-\sum_{\beta\in\widehat{D}_\Sigma^o}\beta$, $D_\Sigma^o{=}D_\Sigma^{(1)}\backslash\{a_{\overline{m}n}\}$. By Lemma 6.11 below, we see that the only possible prime term of $v_2$ is $b_1y_1v_{\Lambda}$, $y_1\in\mathbb{C}$. If $y_1{=}0$ then $v_2$ has no prime term, and by Lemma 5.1(i), $e_{\overline{m}n}v_\Sigma$ has no prime term, which contradicts with Lemma 5.2(i). Therefore $y_1{\ne}0$. Similarly $\widetilde{v}_2$ has one prime term $b_1\widetilde{y}_1v_{\Lambda}$. By rescaling, we can suppose $y_1{=}\widetilde{y}_1$ and then by Lemma 5.1(ii) and Lemma 5.2(ii), we obtain $e_{\overline{m}n}v_\Sigma{=}e_{\overline{m}n}\widetilde{v}_\Sigma$. ∎

*Lemma 6.11.* There is a unique $b_1=\prod_{\beta\in\widehat{D}_\Sigma^o}f(\beta)\in B$ with weight $-\sum_{\beta\in\widehat{D}_\Sigma^o}\beta$.

*Proof.* Using (6.16) as an example will help us to understand the proof. Suppose there exists

another $b_2 = \prod_{\beta\in\widehat{D}_1} f(\beta) \in B$, $D_1 \subset D$ with the same weight $-\gamma$ so that $\gamma = \sum_{\beta\in\widehat{D}_1}\beta = \sum_{\beta\in\widehat{D}^o_\Sigma}\beta$. Let $\gamma = \sum_{i=\overline{m}}^n a_i\alpha_i$ with the coefficients $a_i$, then we have

$$\#\{i|(i,j)\in D_1\} = \#\{i|(i,j)\in D^o_\Sigma\} = a_{j-1}-a_j, \qquad 1\le j\le n+1, \tag{6.21a}$$
$$\#\{j|(i,j)\in D_1\} = \#\{j|(i,j)\in D^o_\Sigma\} = a_{\overline{m+1-i}}-a_{\overline{m+2-i}}, \quad 1\le i\le m+1, \tag{6.21b}$$

where, if $j{=}n{+}1$ or $i{=}1$ we suppose $a_{n+1}{=}a_{\overline{m+1}}{=}0$. It suffices to prove that the solution in (6.21) is $D_1{=}D^o_\Sigma$. If not, suppose $(i_0,j_0){\in}D_1\backslash D^o_\Sigma$ with $j_0$ being the smallest. First, suppose $(i_0,j_0){=}(1,n{+}1)$. Then (6.21a) in the case $j{=}n{+}1$ tells us that there exists $(i_1,n{+}1){\in}D^o_\Sigma\backslash D_1$ with $i_1{\ne}i_0$ and (6.21b) in the case $i{=}i_1$ tells us that there exists $(i_1,j_1){\in}D_1\backslash D^o_\Sigma$ with $j_1{\ne}n{+}1$ and so, $j_1{<}n{+}1{=}j_0$, which contradicts with the choice of $j_0$ being the smallest. Second, suppose $(i_0,j_0){\ne}(1,n{+}1)$. Then by Lemma 6.9(i), we have $i_0{>}x$ or $j_0{<}n{-}x{+}1$. Suppose $j_0{<}n{-}x{+}1$ (the other case is similar). Then using (6.21a) in the case $j{=}j_0$, there exists some $(i_1,j_0){\in}D^o_\Sigma\backslash D_1$. By Lemma 6.9(i) we have $i_1{\le}x$. Also $i_1{\ne}1$ since $D$, and so, $D^o_\Sigma$, does not contain position $(1,j_0)$. Now consider (6.21b) for $i{=}i_1$, from $(i_1,j_0){\in}D^o_\Sigma\backslash D_1$, there exists some $j_1$ such that $(i_1,j_1){\in}D_1\backslash D^o_\Sigma$; however, if $j_0{\le}j_1$, by Lemma 6.9(ii)(a) we would have $(i_1,j_1){\in}D^o_\Sigma$, thus, $j_1{<}j_0$, again contradicting with the choice of $j_0$ being the smallest. ∎

Now we can complete the proof of Theorem 6.6 as follows. By Lemma 6.10, we can suppose $e_{\overline{m}n}(v_\Sigma{-}\widetilde{v}_\Sigma){=}0$. Since $e_{\overline{m}}(v_\Sigma{-}\widetilde{v}_\Sigma){=}e_{\overline{m}}v_\Sigma{\ne}0$, let $k$ be the largest such that $v_\lambda{\equiv}e_{\overline{m}k}(v_\Sigma{-}\widetilde{v}_\Sigma){\ne}0$ with $k{<}n$. As $\widetilde{v}_\Sigma$ is $G^{(m/n-1)}$-primitive, we have $v_\lambda{=}e_{\overline{m}k}v_\Sigma$ with weight $\lambda{=}\alpha_{\overline{m}k}{+}\Sigma$. Applying $e_i$ to $v_\lambda$ and (2.4) gives $e_iv_\lambda{=}0$, $i > \overline{m}$. We also have $e_{\overline{m}}v_\lambda{=}0$: if not, again similar to the arguments after (6.11), let $g_1{\in}\mathbf{U}(G^+)$ with largest possible weight $\mu_1$ such that $u{=}g_1v_\lambda{=}gv_\Sigma{\ne}0$, where now $g{=}g_1e_{\overline{m}k}$ with weight $\mu = \mu_1{+}\alpha_{\overline{m}k}$, then as $\langle\Sigma{+}\rho|\alpha_{\overline{m}k}\rangle > 0$, we could find no solution for $\mu$ (or $\mu_1$) in (6.12). Thus we have in fact proved that $v_\lambda$ is $G$-primitive, which is not possible since $e_{kn}v_\lambda{=}e_{\overline{m}n}v_\Sigma{\ne}0$. The contradiction shows that the assumption (6.19) is wrong, so, either $v_\Sigma$ or $\widetilde{v}_\Sigma$ must be $G$-primitive, proving Theorem 6.6 in the case $TR_D{=}(1,n{+}1)$.

For $TR_D{=}(m{+}1{-}m_\Sigma, n_\Sigma{+}1)$, let $G' = G^{(m_\Sigma/n_\Sigma)}$ and let $U^{(m_\Sigma/n_\Sigma)}$ be the $G'$-submodule of $\overline{V}(\Lambda)$ generated by $v_\Lambda$ isomorphic to $\overline{V}(\Lambda)^{(m_\Sigma/n_\Sigma)}$. Let $\Sigma^{(m_\Sigma/n_\Sigma)}$ correspond to an indecomposable unlinked code $\Sigma^{c(m_\Sigma/n_\Sigma)}$ of $\Lambda$ restricted to $G'$. By construction the topmost and rightmost position of $D_\Sigma^{(m_\Sigma/n_\Sigma)}$ in $A(\Lambda)^{(m_\Sigma/n_\Sigma)}$ is $(1,n_\Sigma+1)$. As just proved, there is a $G'$-primitive vector $v_\Sigma = g_\Sigma v_\Lambda$ corresponding to the code $\Sigma^{c(m_\Sigma/n_\Sigma)}$ with $g_\Sigma\in\mathbf{U}(G'^-)$, which commutes with $e_i$, $i\in\{\overline{m},\overline{m-1},\cdots,\overline{m_\Sigma+1},n_\Sigma{+}1,\cdots,n\}$. Hence $v_\Sigma$ is also $G$-primitive corresponding to the code $\Sigma^c$. This completes the proof of Theorem 6.6 in general. ∎

**Theorem 6.12.** To any unlinked code $\Sigma^c$ for $\Lambda$, there corresponds a primitive vector $v_\Sigma = g_\Sigma v_\Lambda$ of $\overline{V}(\Lambda)$ with weight $\Sigma$ for some $g_\Sigma\in\mathbf{U}(G^-)$.

*Proof.* Suppose $\Sigma^c = \Sigma^c_1\cdots\Sigma^c_k$ with all $\Sigma^c_i$ indecomposable unlinked codes. The proof is covered by Theorem 6.6 if $k=1$. Let $k>1$. By Definition 3.12, we see that $\Sigma^c_k$ is an indecomposable code of $\Lambda$, thus corresponding to a primitive vector $v_{\Sigma_k} = g_{\Sigma_k}v_\Lambda$. We also

see that $\Sigma_o^c = \Sigma_1^c \cdots \Sigma_{k-1}^c$ is an unlinked code for the highest weight vector $v^*_{\Sigma_k}$ of $\overline{V}(\Sigma_k)$ with weight $\Sigma_k$. By induction on $k$, there exists a primitive vector $v^*_{\Sigma_o} = g_{\Sigma_o} v^*_{\Sigma_k}$ in $\overline{V}(\Sigma_k)$. Let $v_{\Sigma} = g_{\Sigma_o} g_{\Sigma_k} v_{\Lambda} \in \overline{V}(\Lambda)$ be the image of $v^*_{\Sigma_o}$ under the homomorphism $\overline{V}(\Sigma_k) \to \overline{V}(\Lambda)$: $v^*_{\Sigma_k} \to v_{\Sigma_k}$, $g v^*_{\Sigma_k} \to g v_{\Sigma_k}$, $g \in \mathbf{U}(G)$. One can check that $v_{\Sigma}$ is nonzero as its leading term up to a non-zero scalar is $\prod_{\beta \in \widehat{D}_{\Sigma}} f(-\beta) v_{\Lambda}$, hence it is a primitive vector corresponding to $\Sigma$. ∎


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